Transmission-Line Properties of Parallel Wide Strips by a Conformal-Mapping Approximation

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Summary—A transmission line is made of a symmetrical pair of strip conductors, face-to-face, or a single strip parallel to a ground plane. The strip width is nominally greater than the separation, but may be somewhat less than the separation.

The field configuration is evaluated by a conformal mapping procedure which gives a very close approximation in terms of ordinary functions (exponential and hyperbolic) rather than the exact solution in terms of difficult functions (elliptic). Computation procedures are given for synthesis or analysis, the former following more naturally from the derivation. For the transitional case of strip width equal to separation, the mapping approximation is found to leave a relative error of the order of 10^{-8}, in the wave resistance or shape ratio. Simple, explicit, practical formulas are developed for practical use with slide-rule accuracy. Emphasis is placed on establishing and specifying the residual error of each formula.

I. Introduction

IN THE DECADE of 1945 to 1955, there was much interest in the subject of electromagnetic wave transmission lines formed of strips of sheet conductor in various configurations. This culminated in a special issue of these TRANSACTIONS in March, 1955.

A substantial part of the effort was devoted to the mathematical computation of the properties of strip lines. Because this was a problem in two-dimensional electric and magnetic fields, it was natural to apply the principles of orthogonal transformations on the plane of the complex variable, a familiar technique known as conformal mapping The resulting formulas were usually so complicated that any practical utility resulted from simplified approximations for limited ranges of the variables.

One of the simplest and most useful configurations was most resistant to solution in useful form. This is the symmetrical case of parallel flat strips, face-to-face, or the corresponding asymmetrical case of one flat-strip parallel to a ground plane. The exact solution has been expressed in implicit form in terms of elliptic functions, [2]–[4], [6], [13]. The most interesting range, in which the strip width is comparable to the separation, has been most resistant to simple derivation in a form convenient for computation and established within some limits of error.

In 1954, the author perceived a procedure for approximate conformal mapping which gave the essential relations in simple forms in terms of "slide-rule" functions (exponential and hyperbolic) [9], [12]. Moreover, the limits of error were clearly established and the closeness of approximation was remarkable. This approach was based on the approximations for "wide" strips, that is, strip width exceeding the separation, but it provided a close approximation for moderately narrow strips. Its range of validity overlapped that of simple formulas based on approximations for "narrow" strips.

The objective of this paper is the presentation of this development in conformal mapping. It is intended to serve a variety of purposes, as follows:

1) To enable the reliable computation of this case, in terms of "slide-rule" functions, close enough for the most exacting requirements of mathematical scrutiny.
2) To yield simple approximate formulas for practical computations to "slide-rule" accuracy, and for showing clearly the principal effects of the variables.
3) To present a method of simple approximation in conformal mapping, applicable to various configurations of "wide" strip conductors, and susceptible of determination within close limits of error. The method is a departure from the straightforward exact procedure.

In the formulation of transmission-line properties, it is required to express a relation between its electrical and structural properties. The most significant electrical property is its wave resistance. The most significant structural property is its shape ratio (strip-width over separation). The conductors are assumed to be very thin sheets of perfectly conductive material. The space is assumed to be filled with a dielectric wave medium free of dissipation.

The classical approach to the formulation of transmission-line properties has been that of analysis. By this is meant the evaluation of electrical properties explicitly in terms of structural dimensions. This is basically an academic approach, as distinguished from the engineering approach which requires the design of a structure to satisfy certain electrical requirements. The latter approach is that of synthesis, which has been on the ascendency over the past few decades. This

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contrast is particularly interesting in the present topic, because the formulas for synthesis come more naturally. They are found to be more susceptible to simplification and close approximation. This is another example of a principle the writer has applied previously in the study of other configurations of transmission lines [5], [8].

There is particular interest in the same configuration of conductors applied to the opposite faces of a dielectric sheet. A simple solution to this problem of mixed dielectric appears not to have been published, and it would be helpful in the practical design of printed lines. The present derivation is intended, incidentally, to provide the background for a solution of this problem to be presented subsequently.

The present application of conformal mapping will first be described with reference to the three significant planes to be involved in the orthogonal transformations. Then the mapping gradients will be specified and the essential mathematical relations derived from them. From these will be derived some incidental relations and useful approximations. Finally there will be given some procedures for computation, ranging from the complete application of the mapping approximation down to some simple practical formulas. These will be organized for either synthesis or analysis. Some of the numerical results will be given in tabular and graphical form. The mathematical symbols, which are introduced in the text, are listed at the end of the paper in Section XII for convenient reference.

II. Conformal Mapping

The principles of conformal mapping are applied here to give a close approximation in simple form, rather than an exact representation in terms of elliptic functions.

Fig. 1 shows the cross section of a transmission line formed of a symmetrical pair of parallel strips. Each strip is assumed to be a perfect conductor of zero thickness. The essential dimensions are the strip width ($2a$) and separation ($2b$). The shape ratio (a/b or b/a) is the significant parameter. The balanced pair has a certain wave resistance (R). The capacitance between the pair of strips is the classic case of a parallel-plate capacitor with an edge effect associated with the "fringing field."

The same solution is applicable to the case of a single strip spaced from a parallel ground plane by the half separation (b) giving half the wave resistance ($\frac{1}{2}R$).

Here the wave resistance of the line is to be expressed relative to the wave resistance (R_c) of the dielectric medium (377 ohms in free space or air).

Fig. 2 shows the space coordinates of one quadrant of the cross section of the balanced pair of strips. The wave resistance of this quadrant is the same as that of the pair (R). The quadrant includes the half width (a) and the half separation (b) on one quadrant of the z plane ($x+jy=z$). The critical points for mapping are designated by numbers (1 to 8). The heavy solid lines represent electric equipotential boundaries (conductors) and the heavy dashed lines represent flux-bisection boundaries. The light dashed line shows the flux line off the edge of the strip (3, 7).

While the emphasis here is on "wide" strips ($a/b>1$), the strips are shown rather narrow (a/b about $\frac{1}{4}$) so that some regions of mapping remain large enough to diagram roughly to scale. This shape is roughly the limit of close approximation in the present derivations for "wide" strips.

At the left of the quadrant of interest, the conductor lines are extended from (4) to ($5''$ at $x=-\infty$) to (6). This region will be excluded in the mapping operation. The half separation is made equal to a half circle of angle ($b=\pi$) to correspond with the associated reversal of direction along the conductor boundary ($jy=j\pi=ln-1$).

Fig. 3 shows the coordinates on which the conductor boundaries are mapped on a straight line. This is the w plane ($u+jv=w$). The mapping is such that changes in direction are limited to reversals, so the mapping gradients are rational fractions. The flux-bisection boundaries (y axis) are not colinear, but join the potential boundaries at right angles, as in Fig. 1.

The excluded area appears inside of a region, (5), which is approximately a semicircle. This is the focal region, which includes a group of quantities which will assume particular significance. Therefore this region is shown enlarged in Fig. 4, to emphasize the small differences of these quantities, as will appear in the mathematical presentation. This region on the w plane corresponds to the left quadrant on the z plane, which is to be excluded in the mapping operation.

If the four quadrants of the w plane are completed by imaging, the excluded regions have nearly the same shape as the cross section of a pair of round wires. This is the idea behind the approximations to be relied on. In this case, however, the flux and potential contours are interchanged, so the w plane is related by duality to the pair of round wires.

Fig. 5 shows the flux-potential coordinates on which the actual boundaries are to be mapped. They are to match the coordinate lines exactly on three of the four sides and approximately on the fourth side. This is the z' plane ($x'+jy'=z'$) of flux (x') and potential (y'). The half separation is the same as on the z plane ($b=\pi$) so that the width is directly comparable with the strip half width. The extra width is a measure of the edge effect. The width is divided in proportion to the flux ratio on the outer and inner faces of the strip, as indicated.

On the fourth side (4 to 6), the departure from a straight line is exaggerated. The first approximation will be based on the ends of this side. The second approximation will be based on the perpendicular straight

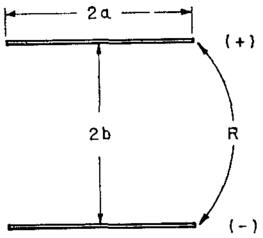

Fig. 1—Transmission line formed of parallel strips.

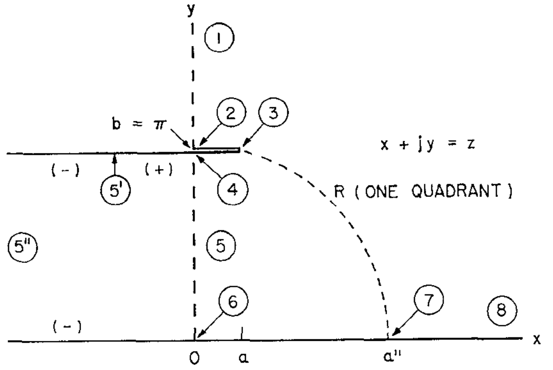

Fig. 2—Space coordinates on z plane.

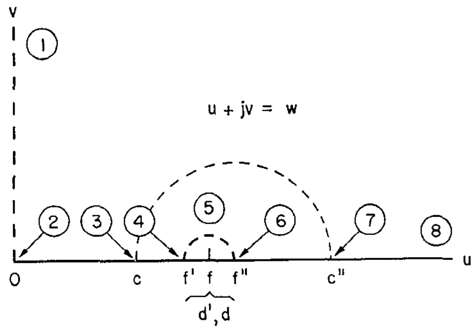

Fig. 3—Colinear-boundary coordinates on w plane.

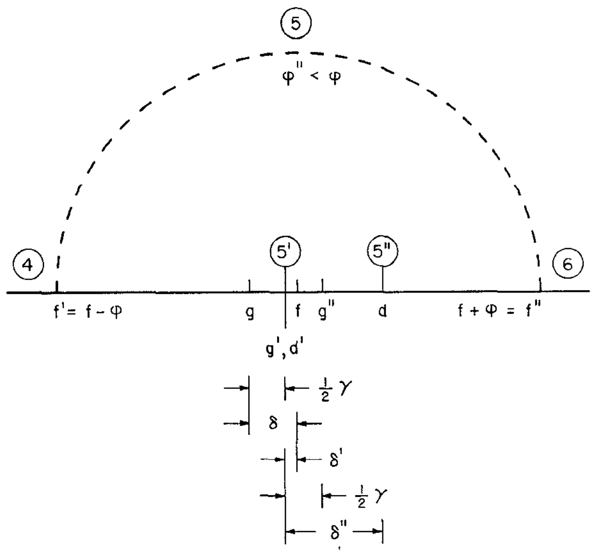

Fig. 4—Enlargement of focal region on w plane.

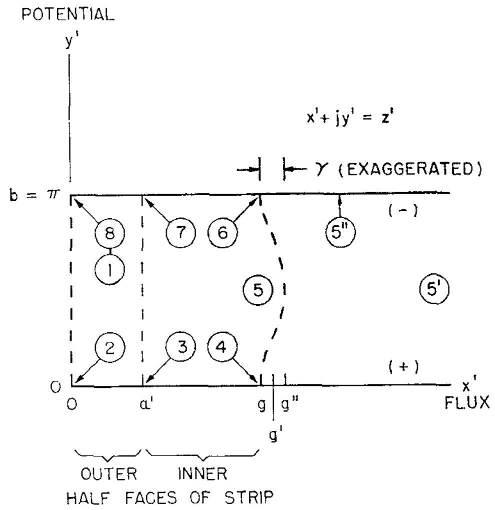

Fig. 5—Flux-potential coordinates on z' plane.

line which includes nearly the same area as the actual boundary, mapped on this plane. The principal curvature of this boundary is associated with the second-order space harmonic as will be implicit in some of the exponential series to be given.

One function of the flux-potential plane is to provide a simple rectangular grid of contours from which the wave resistance of the line becomes apparent. The aspect ratio of the rectangle on the z' plane is nominally g/π, in terms of the effective half width (g). After making the correction for the curvature of one side, the aspect ratio is slightly increased to g'/π. The wave resistance in the quadrant, which is equal to that of the balanced line (Fig. 1), is therefore

$$R = R_c\pi/g'; \qquad g' = \pi R_c/R. \tag{1}$$

(All symbols are listed in a later section.) The corresponding values of inductance (L) and capacitance (C) are related to the length (l) as well as the properties of the dielectric wave medium (μ, ϵ).

$$L = \mu l\pi/g'; \qquad g' = \pi l\mu/L$$

$$C = \epsilon l g'/\pi; \qquad g' = \pi C/\epsilon l.$$

The marginal condition for "wide" strips ($a/b=1$) corresponds roughly to $R=\frac{1}{2}R_c$. Therefore, the case of $g'=2\pi$ is to be taken as representative of this condition. It happens that the present approximation is useful for even narrower strips (say $a/b=\frac{1}{4}$ as diagramed in Fig. 2), corresponding roughly to $R=R_c$. Therefore the case of $g'=\pi$ is to be taken as the extreme condition of narrowest strips for the present study. It will be found that the mapping approximation is still very close at this extreme.

Having introduced the three coordinate planes and the ideas of their relationship, the mapping will be developed.

III. Essential Relations

In the conformal mapping, the first step is to specify the mapping gradient between the z plane and w plane of Figs. 2 and 3. There are two reversals in each quadrant on the z plane, and these are symmetrical about the center on the w plane.

$$\frac{\partial z}{\partial w} = \frac{(w+c)(w-c)}{(w+d)(w-d)} = \frac{w^2-c^2}{w^2-d^2} = 1 + \frac{d^2-c^2}{w^2-d^2} \tag{2}$$

$$\begin{aligned} z &= w - \ln\frac{1+d/w}{1-d/w} = w - 2\,\text{antitanh}\, d/w \\ &= j\pi + w - \ln\frac{1+w/d}{1-w/d} \\ &= j\pi + w - 2\,\text{antitanh}\, w/d. \end{aligned} \tag{3}$$

Here we can give an explicit formulation for z of w but not w of z. The constant of integration is chosen to match the far field on both planes, where the mapping gradient approaches unity.

The second step is to specify the mapping gradient between the z' plane and the w plane of Figs. 5 and 3. There is only one reversal in each quadrant on the z' plane.

$$\frac{\partial z'}{\partial w} = \frac{2d'}{d'^2 - w^2} \tag{4}$$

$$\begin{aligned} z' &= \ln\frac{1+w/d'}{1-w/d'} = 2\,\text{antitanh}\, w/d' \\ &= j\pi + 2\,\text{antitanh}\, d'/w \end{aligned} \tag{5}$$

$$w = d' \tanh \tfrac{1}{2}z'. \tag{6}$$

Here it appears that we can give an explicit formulation for z' of w or w of z'.

From these relations, we can express z of z'.

$$z = j\pi + d' \tanh\tfrac{1}{2}z' - 2\,\text{antitanh}\left(\frac{d'}{d}\tanh\tfrac{1}{2}z'\right). \tag{7}$$

It is noted that d and d' are nearly equal, as seen in Figs. 3 and 4.

Let us locate on all three planes the point (3) which is the edge of the strip.

$$z = j\pi + a; \qquad w = c; \qquad z' = a'. \tag{8}$$

This point is identified by the property $\partial z/\partial w = 0$ with reference to (3).

$$\frac{\partial z}{\partial w} = 1 - \frac{2/d}{1-(w/d)^2}; \qquad 0 = 1 - \frac{2/d}{1-(c/d)^2} \tag{9}$$

$$c = \sqrt{d(d-2)} = \sqrt{(d-1)^2 - 1} < (d-1) \tag{10}$$

$$\begin{aligned} c &= (d-1) - \frac{1}{(d-1) + \sqrt{(d-1)^2-1}} \\ &= (d-1) - \frac{1}{2(d-1)} - \frac{1}{8(d-1)^3} - \cdots \end{aligned} \tag{11}$$

$$(d-1) = \sqrt{c^2+1}; \qquad d = 1 + \sqrt{1+c^2} > 2; \tag{12}$$

also $d > c+1$. From formula (3)

$$a = c - 2\,\text{antitanh}\, c/d < c < (d-1). \tag{13}$$

By (10), utilizing relations of hyperbolic functions,

$$\begin{aligned} 2\,\text{antitanh}\, c/d &= \text{antisinh}\, c = \ln\frac{d+c}{d-c} \\ &= \text{antisinh}\sqrt{(d-1)^2-1} \\ &= \text{anticosh}\,(d-1) \\ &= \ln\left[(d-1) + \sqrt{(d-1)^2-1}\right] \end{aligned} \tag{14}$$

$$a = c - \text{antisinh}\, c. \tag{15}$$

This is the simplest equation for a of c, and can be used for computing c of a by usual methods (trial, interpolation, continuing successive approximation, etc.).

On the z' plane, at point (3),

$$a' = 2\,\text{antitanh}\, c/d' = \ln\frac{d'+c}{d'-c}\,. \tag{16}$$

The next critical points are (4) and (6) in the space between the two strips.

At point (4),

$$z = j\pi; \qquad w = f' = f - \phi; \qquad z' = g. \tag{17}$$

From (3),

$$0 = f' - 2\,\text{antitanh}\, f'/d. \tag{18}$$

At point (6),

$$z = 0; \qquad w = f'' = f + \phi; \qquad z' = g + j\pi. \tag{19}$$

From (3),

$$0 = f'' - 2\,\text{antitanh}\, d/f'' \tag{20}$$

$$f = \tfrac{1}{2}f' + \tfrac{1}{2}f''. \tag{21}$$

From (18) and (20),

$$\tfrac{1}{2}d = \frac{\tfrac{1}{2}f'}{\tanh\tfrac{1}{2}f'} = \tfrac{1}{2}f'' \tanh\tfrac{1}{2}f'' \tag{22}$$

$$\begin{aligned} \frac{f-\phi}{f+\phi} &= f'/f'' = \tanh\tfrac{1}{2}f' \tanh\tfrac{1}{2}f'' \\ &= \frac{\cosh f - \cosh\phi}{\cosh f + \cosh\phi} \quad (Dw\ 653.2) \end{aligned} \tag{23}$$

$$\frac{\phi}{f} = \frac{\cosh\phi}{\cosh f}\,. \tag{24}$$

If d is specified as the parameter to determine the shape, c is given explicitly by (10); f' and f'' are given implicitly by (22). This will be included in one sequence of computation.

At points (4) and (6) on the z' plane and w plane, from (6), (17) and (19),

$$\tanh \tfrac{1}{2}g = f'/d' = d'/f'' \tag{25}$$

$$d' = \sqrt{f'f''} = \sqrt{f^2 - \phi^2} < f \tag{26}$$

$$\frac{f-\phi}{f+\phi} = f'/f'' = \tanh^2 \tfrac{1}{2}g = \frac{\cosh g - 1}{\cosh g + 1} \tag{27}$$

$$\phi/f = 1/\cosh g = \operatorname{sech} g \ll 1 \tag{28}$$

In terms of the pair of circles, this last formula is the shape ratio (radius over separation of centers). It is recognized as the form related to a pair of round wires.

From (24) and (28),

$$\cosh g = \frac{\cosh f}{\cosh \phi}; \qquad g < f. \tag{29}$$

From (22) and (26),

$$\frac{d'}{d} = \left(\frac{\tanh \tfrac{1}{2}f'}{\tanh \tfrac{1}{2}f''}\right)^{1/2} < 1. \tag{30}$$

Here we note a group of quantities in the focal region that are nearly equal (d, d', f, g in Fig. 4) as seen by (21), (29) and (30). Their small differences form the subject of another section.

If the parameter d is specified, all other parameters can be determined directly from the above relations through either explicit or simple implicit formulation. If the shape ($a/b = a/\pi$) is specified, the same is true. The sequence of computation will be discussed further on.

IV. Incidental Relations

There are some relationships that are brought out in the mapping, but which are not essential to the basic computations. One of these is the location of the flux line at the edge of the strip, directly from (3) to (7). The flux inside or outside of this line terminates respectively on the inner or outer face of the strip conductor. In the mapping approximation, this line is taken to be a vertical straight line on the z' plane, Fig. 5. Referring to Fig. 2, it is found that the point (7) extends beyond the edges of the strips by the amount

$$a'' - a = 2 + 2/d - \cdots > 2. \tag{31}$$

In the limit of a very narrow strip, this flux line approaches a circle because the field patterns inside and outside are related by inversion. Therefore, in general,

$$a'' > \pi; \quad a + 2 < a'' < a + \pi; \quad \pi > a'' - a > 2. \tag{32}$$

In words, this flux line extends outward from the edge of the strip to a distance somewhat less than the half separation.

Associated with this flux line is the "flux fraction" here defined, relative to the total flux, as the fraction thereof which terminates on the outer face of the strip conductor. On the same basis, with reference to Fig. 5, this fraction is seen to be a'/g'. It is always less than one half.

For "wide" strips, the limiting value is found to be

$$a'/g' = \frac{b}{\pi a} \ln 2\pi a/b. \tag{33}$$

This fraction is particularly significant in some cases (beyond the scope of this paper) where there are different kinds of dielectric adjacent to the respective faces of the strip conductor.

The flux fraction is related to the classic concept of the "fringing field" or edge correction but is not the same. Here the actual half width of the strip is on the z plane, while the effective half width is g' on the z' plane. The excess of the latter over the former is the "edge correction"

$$\Delta a = g' - a. \tag{34}$$

For "wide" strips, the limiting value may be expressed relative to the half separation.

$$\Delta a/b = \Delta a/\pi = \frac{1}{\pi}(\ln 2g' + 1)$$
$$= \frac{1}{\pi}(\ln 2\pi a/b + 1). \tag{35}$$

In the focal region on the w plane, Fig. 4, there is one point (5′) corresponding to the reversal of direction on the z' plane and a slightly different point (5″) corresponding to the reversal on the z plane. It is interesting to note the location of both of these points on both planes, which will be done to a first approximation. Point (5′) on the z plane, Fig. 2, is located at

$$z = -\ln 1/\phi + j\pi = -(g - \ln 2g) + j\pi. \tag{36}$$

Point (5″) on the z plane, Fig. 5, is located at

$$z' = g + \ln 1/\phi + j\pi = g + (g - \ln 2g) + j\pi. \tag{37}$$

The latter has only mathematical significance. The former has the physical significance of the junction between the opposite conductors of the line as they extend into the space which is to be excluded from the mapping. The location of this junction forces the flux boundary into a vertical position on the z plane, by compensating for the distorting influence of the edge effect.

V. Small Differences

While the foregoing relations are mathematically sufficient for computation of all parameters, they are not the best suited for evaluation of some small differences in the focal region, Fig. 4. Several such differences are defined and evaluated as follows. Each formula is valid to a first approximation, which is all that is significant in the described procedure of conformal mapping.

In each case, the approximate upper limit is indicated for the range of $g>\pi$ or $R/R_c<1$.

From (28) and (29), by eliminating ϕ,

$$\delta = f - g = \tfrac{1}{2}\phi^2 = 2g^2 \exp - 2g < 0.036. \quad (38)$$

From (26),

$$\delta' = f - d' = \delta/f = \phi^2/2f = 2g \exp - 2g < 0.012. \quad (39)$$

From (30),

$$\delta'' = d - d' = 2\delta = \phi^2 = 4g^2 \exp - 2g < 0.073. \quad (40)$$

In these formulas, it is noted that $\phi = 2g \exp - g$ to the first approximation.

Another small difference is related to the mapping approximation, as limited by the departure from a straight line in one boundary (4 to 6) on the z' plane, Fig. 5. The greatest departure is near the center at (5); it is denoted

$$\gamma = g'' - g. \quad (41)$$

The quantity g'' is defined in terms of the point $(z' = g'' + j\pi/2)$ half way between the lower and upper boundaries. It is evaluated to correspond to some point (5) on the boundary on the z plane, Fig. 2, by specifying only one requirement $(x=0)$. It is found that

$$\gamma = 4g(g-1) \exp - 2g \pm (\cdots) g^2(g-1)^2 \exp - 4g \pm \cdots. \quad (42)$$

Since the curvature of the approximate boundary is nearly a sine wave, the average location of the boundary is near the straight line

$$x' = g' = g + \tfrac{1}{2}\gamma. \quad (43)$$

Therefore, an approximate correction for the curvature is obtained by taking the boundary to be located at g', a little beyond the end points at g. From (42),

$$\tfrac{1}{2}\gamma = g' - g = (1 - 1/g)\delta = (1 - 1/g)\tfrac{1}{2}\phi^2 = 2g(g-1) \exp - 2g < 0.025. \quad (44)$$

Some differences of particular interest are

$$d' - g' = \delta - \delta' - \tfrac{1}{2}\gamma = 0\phi^2. \quad (45)$$

Since this difference approaches zero relative to the others, it is seen that g' and d' are the same for present purposes, as indicated in Fig. 4. Then

$$d - g' = d - d' = \delta'' = 4g^2 \exp - 2g = 4d^2 \exp - 2d < 0.073. \quad (46)$$

This difference is particularly useful in shifting between d and g', which are two of the principal parameters.

Here we may consider the residual error of the mapping approximation. On the z' plane, Fig. 5, the effective width is some kind of mean between the limits g and g'', or $g' \pm \frac{1}{2}\gamma$. On the basis of g' or the corresponding value of R, the relative error of the limits is given by (53).

$$\gamma/2g' = 2(g'-1) \exp - 2g' < 0.00004 \quad \text{if } g' > 2\pi; \quad < 0.008 \quad \text{if } g' > \pi. \quad (47)$$

While these limits of error are small, the residual error of the mean is much smaller. It has been estimated on the basis of the field distortion and the uncertainty of the kind of mean value that would be valid. As a result, the relative error of the mean is believed to be less than

$$\tfrac{1}{2}g'(\gamma/2g')^2 = 2g'(g'-1)^2 \exp - 4g' < 4 \times 10^{-9} \quad \text{if } g' > 2\pi; \quad < 0.0001 \quad \text{if } g' > \pi. \quad (48)$$

The extremely small residual error emphasizes the high degree of refinement in the mapping approximation. The range of usefulness extends to somewhat smaller g' (higher R, narrower strips), but the residual error becomes uncertain as $(g-1)$ approaches zero.

Referring to the w plane in Fig. 4, the excluded region (5) departs slightly from a circle. Its vertical radius (ϕ'' corresponding to g'' on the z' plane) is slightly less than its horizontal radius (ϕ corresponding to g) approximately in the following ratio.

$$\phi''/\phi = 1 - 4(g-1)^2 \exp - 2g. \quad (49)$$

This ratio departs from unity by less than 0.034, and this is compensated by using, in effect, the average radius (ϕ' corresponding to g').

One application of small differences is in expressing the residual error in the essential relationship between shape and wave resistance. Since the latter is the functional quantity and the former only a means to an end, the relative error should be expressed for wave resistance, even in formulas for the shape ratio. In this presentation, it happens that the relative error of the wave resistance is always less than that of the shape ratio; conversely, the tolerance of error of the shape ratio is greater than that of the wave resistance.

VI. Simple Approximations

Some of the basic relations are simple. Others are complicated enough to justify an attempt at simplification. The simple forms to be given here are close enough for slide-rule computations in the range of "wide" strips ($a/b>1$; $g>2\pi$; $R>\frac{1}{2}R_c = 188$ ohms). The only need for closer approximation is for extending the range to narrower strips or for critical study of the residual errors.

First, we ignore the differences among g, g', d', f, d. Referring to R, the residual relative error is about

$$\delta''/g = 4g \exp - 2g < 0.0001. \quad (50)$$

The mapping formulas then lead to a simple and instructive relation between the z plane and the z' plane.

$$\frac{\partial z}{\partial z'} = \frac{\partial z}{\partial w}\frac{\partial w}{\partial z'} = \frac{c^2 - w^2}{d^2 - w^2}\frac{c^2 - w^2}{2d} = \frac{c^2 - w^2}{2d} \tag{51}$$

$$w = d \tanh \tfrac{1}{2}z' \tag{52}$$

$$z = j\pi + w - z' \tag{53}$$

$$a' = \operatorname{antisinh} c = \operatorname{anticosh}(d - 1) = \ln 2(d - 1) \tag{54}$$

$$a'/g' = \frac{\ln 2(g - 1)}{g} \tag{55}$$

$$d = 1 + \cosh a' \tag{56}$$

$$c = \sinh a' = (d - 1) - \exp - a'. \tag{57}$$

There is one particularly interesting formula which is based on the principle that a is the maximum value of x with respect to u, so it is noncritical to the value of u. Referring to (3),

$$x = u - \ln \frac{d + u}{d - u} \cdot \tag{58}$$

Let $u = c = d - 1$ (approx.) from (11):

$$a = \max x = (d - 1) - \ln(2d - 1). \tag{59}$$

An expansion of (15) in this form gives

$$a = (d - 1) - \ln(2d - 1) + \frac{1}{8(d - 1)^2} - \cdots, \tag{60}$$

in which the first two terms leave a relative error less than 0.001 of d that is attributable to the approximation leading to (59).

This simple formula (59) can be converted from explicit synthesis to explicit analysis by a continuation which converges rapidly. Then this continuation can be terminated to give the correct value for $d = 2\pi$.

$$\begin{aligned}(2d - 1) &= 2a + 1 + 2\ln(2d - 1)\\ &= 2a + 1 + 2\ln[2a + 1 + 2\ln(2d - 1)]\\ &= 2a + 1 + 2\ln\left[2a + 1 + 2\ln(4\pi - 1) + 2\ln\frac{2d - 1}{4\pi - 1}\right]\\ &= 2a + 1 + 2\ln[2a + 1 + 2\ln(4\pi - 1)] \pm \cdots \end{aligned} \tag{61}$$

$$\begin{aligned} g' = d &= a + 1 + \ln[2a + 1 + 2\ln(4\pi - 1)]\\ &= a + 1 + \ln 2\pi(a/\pi + 0.94). \end{aligned} \tag{62}$$

(Later, 0.94 is changed to 0.92 to make the formula nearly exact for $g' = 2\pi$.) The loss of the residual term leaves a relative error less than

$$\frac{4}{\epsilon^2(4\pi - 1)^2} = 0.004. \tag{63}$$

The edge effect is given by $(d - a)$ in terms of the shape parameter (a).

While the present study is directed to thin strips, a moderate thickness can be compensated by a reduction of width. While retaining the same separation $(2b)$, each strip is taken to have a rectangular cross section of small thickness $(\Delta b \ll b)$ and a reduced width $(2a = 2\Delta a)$. Each edge recedes by a small amount $(\Delta a \ll a)$. In the limit of wide strips $(a/b \gg 1)$, the following equation gives the relation for retaining the same properties as thin strips of full width $(2a)$.

$$\Delta a = \frac{\Delta b}{2\pi}\left(\ln \frac{2b}{\Delta b} + 1\right). \tag{64}$$

This relation enables an estimate and an approximate adjustment for the edge effect of thickness.

The present study is intended to cover the range of "wide" strips, and to bridge the transition range between wide and narrow strips. The range of "narrow" strips is the subject of another set of equations developed by the author, which will be stated here in order to complement the formulas derived herein. A formula for synthesis and one for analysis are given to a second approximation obtained by the first two terms of a converging series.

$$b/a = \tfrac{1}{4}\exp \pi R/R_c - \tfrac{1}{2}\exp - \pi R/R_c + \cdots. \tag{65}$$

If $b/a > 1$, relative error < 0.005 of R,

$$R/R_c = \frac{1}{\pi}\ln 4b/a + \frac{1}{8\pi}(a/b)^2 - \cdots. \tag{66}$$

If $b/a > 1$, relative error < 0.02.

Here, the first term is well known but the second term is believed to be new. The residual relative error is proportional to $(a/b)^4$, so it becomes negligible for strips somewhat narrower than the "square" shape, say $a/b < \frac{1}{2}$.

VII. Computation Procedure for Synthesis

For design of a transmission line, the less usual but more logical formula is one giving the shape ratio in terms of the required wave resistance. There are cases, including the present topic, where this approach from the viewpoint of synthesis leads to simpler formulas. (Another example is [8].) First there will be given a procedure for synthesis based on the complete application of the mapping approximations here presented. This procedure (A) retains all the accuracy of the present approximations over the intended range of validity $(R < R_c)$. A second procedure (B) will be given, which is applicable to "wide" strips $(R < \frac{1}{2}R_c)$ with slide-rule accuracy.

A. Close Computation for Synthesis

Specify

$$R/R_c < 1.$$

From (1) and (45),

$$g' = d' = \pi R_c/R.$$

From (46) and (1),

$$\begin{aligned} d &= d' + \delta'' \\ &= \pi R_c/R + (2\pi R_c/R)^2 \exp - 2\pi R_c/R > \pi. \end{aligned} \tag{67}$$

From (13),

$$a/b = \frac{1}{\pi}\sqrt{d(d-2)} - \frac{1}{\pi}\operatorname{anticosh}(d-1). \tag{68}$$

From (47) and (48), relative error of R.
From (10),

$$c = \sqrt{d(d-2)}.$$

From (16),

$$a' = \ln \frac{d'-c}{d'+c}\cdot$$

Flux fraction,

$$a'/g'.$$

B. Simple Computation for Synthesis

Specify

$$R/R_c < \tfrac{1}{2}; \qquad d = \pi R_c/R.$$

From (59),

$$a/b = R_c/R - \frac{1}{\pi}\left[\ln(2\pi R_c/R - 1) + 1\right] > 0.90. \tag{69}$$

Relative error <0.002 of a/b.
Relative error <0.001 of R.
In this formula, the second term represents the edge effect.

$$\text{Flux fraction, } a'/g' = \frac{R}{\pi R_c}\operatorname{anticosh}(\pi R_c/R - 1). \tag{70}$$

VIII. Computation Procedure for Analysis

Naturally the foregoing procedures for synthesis can be applied in reverse as implicit solutions for the wave resistance (R) from the shape ratio (a/b). However, there are different procedures that involve implicit solutions only in certain steps, so they are preferable for analysis. These are summarized below.

A. Close Computation for Analysis

Specify

$$a/b > 0.17; \qquad a = \pi a/b.$$

From (15), compute c (implicit).
From (12),

$$d = 1 + \sqrt{1 + c^2}.$$

From (26), compute δ''.
From (45),

$$g' = d - \delta'' > \pi.$$

From (1),

$$R = R_c\pi/g' < R_c$$

From (47) and (48), relative error.

B. Simple Computation for Analysis

The simple procedure for synthesis gives a simple implicit solution for analysis. Nearly as close an approximation in the form of an explicit solution for analysis is given for whatever interest and utility it may offer.

Specify

$$a/b > 0.90.$$

From (62),

$$R/R_c = \frac{1}{a/b + \frac{1}{\pi}\ln 2\pi\epsilon(a/b + 0.92)} < \frac{1}{2}, \tag{71}$$

in which 0.92 is chosen to give the correct relation for $R/R_c = \frac{1}{2}$; this is a slight departure from (62). Relative error <0.004. This formula has been given previously by the writer (see [9]) and similar forms of lesser accuracy have been given by other writers, *e.g.* [7].

IX. Computed Examples

Table I and Fig. 6 give numerical examples computed from the formulas given here. The table is based on careful slide-rule computations, so the residual relative error may be about 0.002. As much of the complete procedure was used as was justified for slide-rule computation. Most of the examples are in a binary series of wave resistance (R, g').

Fig. 6 is a graph of the relation between the shape ratio (b/a) as abscissas and the wave resistance (R) or flux fraction (a'/g') as ordinates plotted on log-log coordinates. Two principles appear on the graphs; one is the increasing edge effect with increasing separation and the other is the flux fraction approaching $\frac{1}{2}$ with increasing separation. Both graphs extend into the region of large shape ratio (b/a) where close approximate formulas can be derived on the basis of "narrow" strips.

TABLE I
COMPUTED EXAMPLES

R	R/R_c	g'	a	a/b	b/a	a'	a'/g'
377.	1	π	0.545	0.1735	5.76	1.477	0.470
251.	2/3	$\frac{3}{2}\pi$	1.592	0.506	1.972	1.992	0.423
188.5	1/2	2π	2.840	0.904	1.106	2.348	0.373
125.7	1/3	3π	5.545	1.765	0.566	2.821	0.300
94.2	1/4	4π	8.394	2.672	0.374	3.139	0.250
75.4	1/5	5π	11.26	3.58	0.279	3.38	0.215
47.1	1/8	8π	20.24	6.45	0.155	3.875	0.154
23.6	1/16	16π	44.63	14.2	0.0704	4.59	0.091
178.	0.4725	6.65	π	1.	1.	2.416	0.363

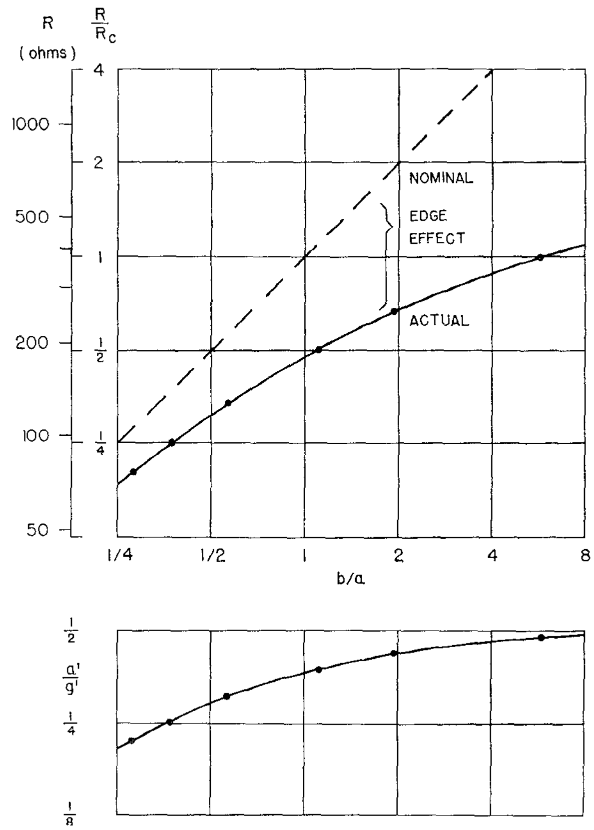

Fig. 6—Graph of wave resistance and flux fraction.

XII. SYMBOLS

R_c = wave resistance of square area of dielectric medium.

R_c = 377 ohms in free space.

R = wave resistance of symmetrical pair of strips or of one quadrant of its cross section.

$x+jy'=z$ = complex plane of space coordinates, the cross section of the pair of strips.

$x'+jy'=z'$ = complex plane of flux-potential coordinates.

$u+jv=w$ = complex plane of colinear boundary coordinates.

a = half-width of strip conductor.

b = half-separation of parallel strips.

a/b = shape ratio.

$c=u$ corresponding to edge of strip.

a' = effective half width of outer face of strip.

g' = mean effective half width of strip, including flux on both outer and inner faces.

g'/π = ratio of effective width over separation.

a'/g' = fraction of total flux that terminates on outer face of strip.

ϕ, ϕ'' = maximum and minimum radii of exclusion circle on w plane.

a, a'' (see Fig. 2).

c, c'' (see Fig. 3).

d, d', f, f', f'' (see Figs. 3, 4).

ϕ, ϕ'', δ, δ', δ'' (see Fig. 4).

g, g', g'', γ (see Figs. 4, 5).

a' (see Fig. 5).

$\epsilon = 2.718$ = base of natural logarithms.

X. CONCLUSION

By utilizing an unusual procedure in conformal mapping, there have been developed various relations in the field of a pair of parallel strips used as a transmission line for electromagnetic waves. These relations are based on the assumption of "wide" strips, but the resulting formulas overlap the region of "narrow" strips. The relation between wave resistance and shape ratio is formulated to an extremely close approximation. Other relations are also formulated, such as the flux fraction for the outer faces. This versatility results from the approximation in terms of "slide-rule" functions rather than the exact formulation in terms of elliptic functions.

XI. ACKNOWLEDGMENT

This development is an application of the technique of conformal mapping, which is within the theory of functions of the complex variable. The essential behavior of the derived formulas has been appreciated, [7], but the approximate solutions have been rather crude and the exact solutions rather difficult. It is interesting to note that the principal feature of the subject approximation was developed independently by Dr. Seymour Cohn, then at Stanford Research Institute, and was communicated to the author some time ago [12]. The completion of this presentation has been stimulated by the needs of a printed-circuit array antenna being designed for Hazeltine Corporation. The author is grateful to Mrs. Nancy M. Staron for the careful typing of the extensive mathematical content of this manuscript.

References

[1] H. B. Dwight, "Tables of Integrals," The Macmillan Company, New York, N. Y.; 1934.
[2] A. Betz, "Konforme Abbildung," Springer-Verlag, Berlin, Germany; 1948. (Parallel strips, pp. 241–243; derivation in terms of elliptic functions.)
[3] F. Oberhettinger and W. Magnus, "Anwendung der Elliptischen Funktionen in Physik und Technik," Springer-Verlag, Berlin, Germany; 1949. (Parallel strips, pp. 63–64; exact solution in elliptic functions, graph for low-resistance range.)
[4] E. Weber, "Electromagnetic Fields—Theory and Applications—Mapping of Fields," John Wiley and Sons, Inc., New York, N. Y., Vol. I; 1950. (Parallel plates, pp. 333–338; 356–357; other references.)
[5] H. A. Wheeler, "The Capacitance of Two Parallel Wires of Different Diameter," Wheeler Labs., Inc., Great Neck, N. Y., Monograph No. 14; 1950. (Unusual formulas for balanced pair.)
[6] H. Kober, "Dictionary of Conformal Representations," Dover Publications, Inc., New York, N. Y.; 1952. (Parallel strips, pp. 190, 194–195; exact solution in elliptic functions.)
[7] F. Assadourian and E. Rimai, "Simplified theory of microstrip transmission systems," Proc. IRE, vol. 40, pp. 1651–1657; December, 1952. (Approximate formula for wide strip above a plane; error about 20 per cent for shape corresponding to parallel strips of equal width and separation.)
[8] H. A. Wheeler, "The Transmission-Line Properties of a Round Wire Between Parallel Planes," Wheeler Labs., Inc., Great Neck, N. Y., Monograph No. 19; 1954. Also. IRE Trans. on Antennas and Propagation, vol. AP-3, pp. 203–207; October, 1955. (Formulas for synthesis in terms of exponential series.)
[9] ——, "Handbook—Lines and Guides," Wheeler Labs., Great Neck, N. Y., Rept. No. 522, pp. 9–10; December, 1954. (Similar formulas for parallel wide strips.)
[10] K. G. Black and T. J. Higgins, "Rigorous determination of the parameters of microstrip transmission lines," IRE Trans. on Microwave Theory and Techniques, vol. MTT-3, pp. 93–113; March, 1955. (Many references.)
[11] H. Meinke and F. W. Grundlach, "Taschenbuch der Hochfrequenztechnik," Springer-Verlag, Berlin, Germany; 1956. (Strip lines, p. 174.)
[12] S. Cohn, private communications; May–June, 1958. (Derivation equivalent to formula (81) in this paper.)
[13] W. von Koppenfels and F. Stallman, "Praxis der Konformen Abbildung," Springer-Verlag, Berlin, Germany; 1959. (Parallel strips in general, pp. 340–344; elliptic functions.)
[14] H. P. Kunzi, "Quasikonforme Abbildungen," Springer-Verlag, Berlin, Germany; 1960.
[15] K. S. Packard and R. V. Lowman, "Transmission lines and waveguides," in "Antenna Engineering Handbook," McGraw-Hill Book Company, Inc., New York, N. Y., ch. 30; 1961. (Strip lines, Fig. 30-7; references.)
[16] A. F. Harvey, "Microwave Engineering," Academic Press, London England; 1963. (Strip lines, pp. 407–415; references.)

Realizable Limits of Error for Dissipationless Attenuators in Mismatched Systems

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Summary—**A tutorial exposition for the exact physical error limits due to mismatch for dissipationless attenuators is presented. The results given yield smaller errors than previous existing formulas due to the inclusion of the physical realizability constraint of passivity. Graphs are included for rapidly determining the largest error for a prescribed set of conditions. This work is based on an analysis prepared by D. C. Youla which had a limited circulation.[1]**

Introduction

FORMULAS for the errors resulting from mismatched generator and detector sections in the measurement of the attenuation of a single attenuator are well known.[2,3] However, with a single exception[4] none of these formulas takes into account the phase restrictions on the various coefficients due to the physical realizability constraint of passivity. The usual formulas exhibit limits obtained by choosing the worst possible phase combinations and, therefore, lead to unnecessarily large errors. In this paper a complete solution is presented for the *physical* error limits due to mismatch for *dissipationless* attenuators calibrated under the standard condition of *conjugate termination*. This includes, of course, the class of equal-resistance attenuators but is more general.

General Background

In a previous publication,[5] a complete scattering description for a linear time-invariant $2N$ terminal net-

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[1] D. C. Youla, "Limits of Errors in Mismatched Attenuators—Part I," Polytechnic Institute of Brooklyn, Brooklyn, N. Y., Memo. No. 68, PIBMRI-1042-62.

[2] C. G. Montgomery, "Technique of Microwave Measurements," McGraw-Hill Book Co., Inc., New York, N. Y., p. 824; 1947.

[3] R. W. Beatty, "Mismatch Errors in the Measurement of Ultrahigh Frequency and Microwave Variable Attenuators," NBS Res. Paper 2465, vol. 52, no. 1; January, 1954.

[4] L. O. Sweet and M. Sucher, "The available power of a matched generator from the measured load power in the presence of small dissipation and mismatch of the connecting network," IRE Trans. on Microwave Theory and Techniques, vol. MTT-5, pp. 167–168; April, 1957.

[5] D. C. Youla, "On scattering matrices normalized to complex port numbers," Proc. IRE, vol. 49, p. 1221; July, 1961.