Relativity and electricity

Electromagnetics is usually developed from a sequence of experimentally based postulates. Here special relativity is used to formulate a complete electromagnetic theory from the inverse-square law, thus deepening our understanding of the unity of electric and magnetic fields

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Several facets of the special theory of relativity are reviewed, commencing with the Lorentz equations and culminating in the force transformation law. The basic equations of electrostatic theory are then formulated, after which a relativistic transformation of Coulomb's law is undertaken, the result being the Lorentz force law. Time-varying electric and magnetic fields are defined as constituent terms of the Lorentz force expression and are then shown to satisfy Maxwell's equations. The development includes derivations of the relativistic transformation laws for sources and fields.

The special theory of relativity is concerned with the comparison of physical phenomena as they appear to two observers who are in motion with respect to each other at a constant relative velocity. In his first paper on this subject in 1905, Einstein[1] accepted the principle of relativity* and proposed as a second postulate that light always propagates in empty space with a definite velocity c, which is independent of the state of motion of the emitting body. Using this second postulate, he introduced a technique for synchronizing spatially separated stationary clocks and then showed that two observers in relative motion disagree in their measurements of time and distance intervals. The Lorentz equations were found to provide the proper connection between the spatial and temporal coordinate values each observer would assign to a given event. From this point, Einstein proceeded to show that Maxwell's equations were covariant under a Lorentz transformation if the electric and magnetic fields of the two observers were related through a certain bilinear transformation. Interpreting this transformation, he remarked "that electric and magnetic forces do not exist independently of the state of motion of the system of coordinates."

In 1912, Leigh Page followed up this observation by demonstrating that one could start from Coulomb's law and use the results of special relativity to derive the fundamental relations of magnetostatics.[2] He also exhibited the expression for the induced electromotive force in one wire due to a variation of current in another. This approach was later embodied in a book coauthored with Adams.[3]

In this article a development conceptually akin to that used by Page, but differing from it substantially in detail, will be presented. After reviewing the necessary aspects of special relativity and electrostatics, a direct derivation of the Lorentz force law and Maxwell's equations will be offered. This approach has the advantage of demonstrating that the fields contained in the Lorentz force expression are synonymous with those contained in Maxwell's equations. This conclusion cannot be reached by a conventional development that postulates separate experimental laws for electrostatics, magnetostatics, and electromagnetics. Further satisfaction results from recognition of the fact that, with the aid of special relativity, *all* the laws of electricity, including the Biot–Savart law and Faraday's EMF law, are derivable from a single experimental postulate based on Coulomb's law.

The development will be confined to the case of electric sources in free space, but it is easily extendable to the case in which materials are present.[4] Wherever specific units are needed, the rationalized MKS system will be used.

The Lorentz transformation

Consider two Cartesian-coordinate systems XYZ and $X'Y'Z'$. As suggested by Fig. 1, the respective axes of these two systems are aligned, with the X and X' axes sliding along each other such that X' is moving in the $+X$ direction at a constant speed u. Let an observer O, who is stationary in XYZ, select his time reference so that $t = 0$ corresponds to the coincidence of the origins of XYZ and $X'Y'Z'$. Similarly, let an observer O', who is stationary in $X'Y'Z'$, select his time reference so that

*The principle of relativity, even then, was an old idea, which had earlier gained the support of Newton, among others. Simply stated, it expresses the belief that all the laws of physics are the same everywhere in the universe. In Einstein's words, "... the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good."

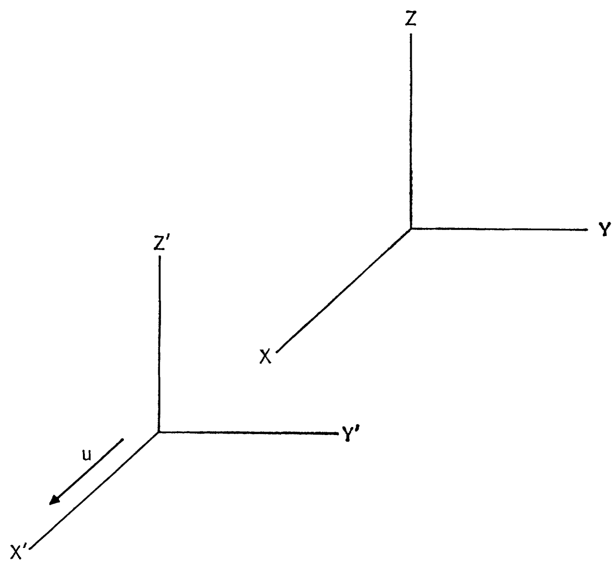


Fig. 1. Two Cartesian-coordinate systems in constant translative relative motion.

$t' = 0$ corresponds to the coincidence of the origins of the two coordinate systems. It is desired to compare the physical observations of O and O'.

Since all physical measurements involve, fundamentally, determinations of distance and/or time intervals, it is necessary to define the concepts of length and time for each observer. To this end, let it be assumed that O determines a unique triplet of numbers (x, y, z) for every point in XYZ space by laying out identical scales (e.g., in meters) along his three axes, and that similarly O' determines a unique triplet of numbers (x', y', z') for every point in $X'Y'Z'$ space by laying out identical scales along *his* three axes. It is further assumed that O and O' lay out these scales using the same standard of length (e.g., a meter stick). By this it is meant that if O measures lengths in terms of a ruler R marked in meters and at rest in XYZ, and if O' measures lengths in terms of a ruler R' at rest in $X'Y'Z'$, then if the two rulers were brought to rest side by side, markings one meter apart on R would coincide with markings one meter apart on R'.

Additionally, each observer, O and O', needs to measure time unambiguously at every point in his coordinate system. To insure this, let it be assumed conceptually that O has an inexhaustible supply of identical clocks, such that he has been able to station one clock permanently at each point in XYZ. To ascertain that all of these clocks are set properly and running at the same rate, O can select one clock as the reference and perform the following experiment: O places himself at the reference clock and stations an auxiliary observer O_1 at the clock to be synchronized. O sends out a pulse of light at time t_a on the reference clock, directing it toward O_1, who reflects it back by means of a mirror. The returned pulse of light reaches O at time t_b. The clock where O_1 is stationed was set properly if it read $(t_a + t_b)/2$ at the instant the light pulse reached the mirror. It is running at the proper rate if it proves to be set properly every time O and O_1 choose to perform this experiment.

In this manner, every clock in XYZ can be synchronized to the reference clock and thus to every other clock in XYZ. It will be assumed that this has been done; this will be the conception of time in the frame of reference XYZ.

Likewise, it may be assumed conceptually that O' has an inexhaustible supply of identical clocks, which he has arrayed at fixed points in $X'Y'Z'$ and which he has synchronized by the same procedure. It will be further assumed that if these two sets of clocks were brought to rest relative to each other, they would be found to be identical and running at the same rate.

With these concepts of spatial position and time, let an event be defined for observer O as something that happens at a point $P(x, y, z)$ at time t, or more briefly at the "point" $P(x, y, z, t)$. The same event will occur for observer O' at the "point" $P'(x', y', z', t')$. The quartet of numbers x', y', z', t' can be connected to the quartet of numbers x, y, z, t by a set of equations known as the Lorentz transformation equations. This transformation may be determined as follows:

Let a pulse of light be emitted from the position jointly occupied by the two origins at the instant the two clocks at this position register $t = 0$, $t' = 0$. Imagine that O has stationed an auxiliary observer O_1 at the fixed point (x, y, z) and that O_1 records the event that the light pulse passes him as having occurred at time t. Then it follows that, if the region is free space, O can characterize this event by the equation

$$x^2 + y^2 + z^2 = (ct)^2 \tag{1}$$

The left side of this equation is the square of the distance from the origin of XYZ to the position of O_1. The right side is the square of the distance a light wave will travel at speed c in empty space in a time interval t. The equation itself is recognized as depicting a spherical wavefront of steadily expanding radius.

If O' has stationed an auxiliary observer O_1' at the *fixed* point (x', y', z'), then the time of occurrence t_1', which O_1' records for this event, satisfies the equation

$$(x')^2 + (y')^2 + (z')^2 = (ct')^2 \tag{2}$$

O' uses the same value for the speed of light in (2) that O uses in (1) because the region is empty space; at most they disagree about the motion of the source, and c has the same value in all directions in $X'Y'Z'$ that it does in all directions in XYZ (Einstein's second postulate).

If O_1 and O_1' just happen to coincide at the instant the light pulse passes, the transformation equations that link the observations in XYZ to those in $X'Y'Z'$ must be such that O can derive (2) from (1) and such that O' can derive (1) from (2), since they are describing the same event. The two observers agree about distance measurements in the Y and Z directions because their relative motion is X-directed. Therefore, part of the transformation is

$$y' = y \qquad z' = z \tag{3}$$

Since every motion that is uniform and rectilinear in XYZ must also appear uniform and rectilinear in $X'Y'Z'$, so that the transformation from (x, t) to (x', t') takes straight lines into straight lines, and is thus *linear*, then the remainder of the transformation must be in the form

$$x' = \alpha_1 x + \alpha_2 t \qquad t' = \alpha_3 x + \alpha_4 t \tag{4}$$

To evaluate the constant α_2, note first that if a point $P'(x', y', z')$ is *fixed* with respect to observer O', this point appears to be moving in the positive X direction at speed u, when observed by O. For such a point, taking differentials of the first equation of (4) gives

$$dx' = 0 = \alpha_1\,dx + \alpha_2\,dt \qquad \alpha_2 = -\alpha_1\frac{dx}{dt} = -\alpha_1 u$$

and therefore (4) may be rewritten as

$$x' = \alpha_1(x - ut) \qquad t' = \alpha_3 x + \alpha_4 t \tag{5}$$

The remaining three constants can be determined by requiring that (1) and (2) transform into each other. If (3) and (5) are substituted into (2), one obtains

$$\alpha_1^2x^2 - 2\alpha_1^2uxt + \alpha_1^2u^2t^2 + y^2 + z^2 = \alpha_3^2c^2x^2 + 2\alpha_3\alpha_4c^2xt + \alpha_4^2c^2t^2$$

Since this must agree with (1) for all values of x, y, z, and t, it follows that

$$\alpha_1^2 - \alpha_3^2c^2 = 1 \qquad 2\alpha_1^2u + 2\alpha_3\alpha_4c^2 = 0$$
$$\alpha_4^2c^2 - \alpha_1^2u^2 = c^2$$

Solving these three equations gives

$$\alpha_1^2 = \alpha_4^2 = (1 - u^2/c^2)^{-1} \qquad \alpha_3 = -\alpha_1u/c^2$$

which yields the result

$$\begin{aligned} x' &= \kappa(x - ut) \qquad & t' &= \kappa(t - ux/c^2) \\ y' &= y & z' &= z \end{aligned} \tag{6}$$

in which $\kappa^{-1} = (1 - u^2/c^2)^{1/2}$ is called the contraction factor.

Equations (6) were derived by Einstein in his 1905 paper using an argument which has been reproduced in its essentials. They are commonly called the Lorentz transformation equations, so named by Poincaré in honor of H. Lorentz, who had derived them earlier (1903) under a different set of hypotheses.* Their significance lies in the fact that they may be employed to deduce the four numbers either observer uses to characterize an event, if the four numbers used by the other observer are known.

Length and time under the Lorentz transformation

Let a ruler R' be at rest in the $X'Y'Z'$ frame of reference, such that its two ends occupy the points $(x_1', 0, 0)$ and $(x_2', 0, 0)$. Observer O' will say that its length is

$$l'_{R'} = x_2' - x_1'$$

If observer O wishes to measure the length of R', since it is in motion with respect to him, he should measure its end coordinates x_1 and x_2 at a common time t. Using the first equation of (6), one may write

$$x_1' = \kappa(x_1 - ut) \qquad x_2' = \kappa(x_2 - ut)$$
$$x_2' - x_1' = \kappa(x_2 - x_1)$$

*These equations had actually been used even earlier by Voigt (1887). Lorentz assumed the existence of an ether and physical contraction of bodies due to their motion through the ether, and required that Maxwell's equations transform properly. His ether-related hypotheses were found to be inconsistent with the Michelson–Morley and Kennedy–Thorndike experiments.

from which

$$l_{R'} = x_2 - x_1 = l'_{R'}(1 - u^2c^2)^{1/2} < l'_{R'} \tag{7}$$

in which $l_{R'}$ is the length of the ruler R', as determined by O, and is seen to be shorter than the rest length $l'_{R'}$. If the ruler had been oriented parallel to the Y' or Z' axis, a similar calculation would reveal that O and O' agreed about the length of R'. One concludes from this that when a body is in motion relative to an observer O, its longitudinal dimension is shortened by the contraction factor, whereas its transverse dimensions are unaltered from their rest values.

Next consider a particular clock in XYZ that remains at fixed coordinates (x, y, z) and is therefore being passed by a sequence of $X'Y'Z'$ clocks. One can define a first event when the hands of this single XYZ clock indicate time t_1 and a second event when they indicate time t_2.

In $X'Y'Z'$, the first event will occur at the spatial position

$$x_1' = \kappa(x - ut_1) \qquad y_1' = y \qquad z_1' = z$$

these equations resulting from an application of (6). The $X'Y'Z'$ clock at this position registers the time of the first event as

$$t_1' = \kappa\,(t_1 - ux/c^2)$$

Similarly, in $X'Y'Z'$ the second event will occur at the spatial position

$$x_2' = \kappa(x - ut_2) \qquad y_2' = y \qquad z_2' = z$$

and the $X'Y'Z'$ clock at this position registers its time as

$$t_2' = \kappa\,(t_2 - ux/c^2)$$

From this it follows that

$$t_2' - t_1' = \kappa(t_2 - t_1) > t_2 - t_1 \tag{8}$$

Consider this result first from the viewpoint of O, who is stationary beside the single XYZ clock. He watches a succession of $X'Y'Z'$ clocks go by and can take only a single reading of each of them. However, he notices that they seem progressively set further and further ahead, thus accounting for the inequality in (8). On the other hand, observer O' can take a sequence of readings of the XYZ clock as it passes a succession of $X'Y'Z'$ clocks. Since he knows his own clocks are all synchronized, he concludes that the rate of the XYZ clock is slowed by its relative motion.

The results (7) and (8) are known as length contraction and time dilatation. They are symmetrical, in the sense that either O or O' will conclude that longitudinal lengths in the other system are shortened and that clocks in the other system are slowed. Experimental evidence to support these formulas is abundant.[5]

Velocity

The general motion of a point, in which the spatial variables are continuous functions of the temporal variable, may be traced in terms of differentials. From (6),

$$dx' = \kappa(dx - u\,dt) \qquad dt' = \kappa\left(dt - \frac{u}{c^2}dx\right)$$
$$dy' = dy \qquad dz' = dz$$

Ratios of these differentials may be formed to yield velocity components. For example,

(Continued on page 147)

$$v_x' = \frac{dx'}{dt'} = \frac{dx - u\,dt}{dt - (u/c^2)\,dx} = \frac{dx/dt - u}{1 - (u/c^2)\,dx/dt} = \frac{v_x - u}{1 - uv_x/c^2}$$

Proceeding in this manner, one can derive the Lorentz velocity transformation equations:

$$v_x' = \frac{v_x - u}{1 - uv_x/c^2} \qquad v_y' = \frac{v_y}{\kappa(1 - uv_x/c^2)} \tag{9}$$
$$v_z' = \frac{v_z}{\kappa(1 - uv_x/c^2)}$$

As an illustration of this result, consider the case of two particles moving along the X axis. As seen from the XYZ frame of reference, let one particle have a velocity $v_x = v$ and let the other particle have a velocity $v_x = -v$. What is the relative velocity?

To answer this question, let $X'Y'Z'$ ride along with one particle by setting $u = v$. Then from (9), the velocity of the other particle in $X'Y'Z'$ is

$$v_x' = \frac{v_x - u}{1 - uv_x/c^2} = \frac{-v - v}{1 + v^2/c^2} = -\frac{2v}{1 + v^2/c^2}$$

For small values of v/c this yields the classic result $v_x' = -2v$. However, as $v \to c$, $v_x' \to -c$. Therefore, even though in XYZ the two particles might be going in opposite directions, each of which approaches c relative to XYZ, their recessional velocities relative to each other are still less than c.

The variation of mass

A hypothetical experiment first suggested by Lewis and Tolman[6] serves to demonstrate the dependence of mass on relative velocity. Imagine that two exactly similar elastic balls suffer a collision, which in the $X'Y'Z'$ frame appears as shown in Fig. 2(A). The balls are seen to approach each other along parallel lines, collide, and then recede from each other along parallel lines. Their approach speeds are equal and, by symmetry, so too are their recessional speeds. A perfectly elastic collision is assumed, with no loss of energy, thus causing the recessional speed to equal the speed of approach. This experiment can be assumed to take place either in a region free from gravitational attraction or on a level frictionless table over which the balls are sliding.*

Now imagine this same collision as viewed from an XYZ frame that is moving in the direction of the $-X'$ axis at a speed $u = v_x'$; see Fig. 2(B). To an observer O stationary in XYZ, ball A is moving parallel to the Y axis and ball B makes a more grazing incidence to the X axis.

As seen in $X'Y'Z'$, each ball has its y' velocity component reversed by the collision, but its x' component of velocity is unchanged. As seen in XYZ, ball B has its y component of velocity reversed by the collision; however, its x component is unaffected. In XYZ, ball A has only a y velocity component, which suffers a reversal.

Classical mechanics would yield for this experiment the result that $v_y = v_y'$ for both balls A and B. In terms of a Lorentz transformation, one would be ill-advised to assume this result without checking. Therefore, let $\pm w_y$ represent the velocity of ball A in XYZ before and after the collision, and let $\mp v_y$ represent the y component of velocity of ball B before and after the collision.

*A rolling motion would complicate the discussion needlessly.

Using (9), one finds that for ball B

$$v_y' = \frac{v_y}{\kappa(1 - v_x'v_x/c^2)}$$

whereas for ball A

$$v_y' = w_y/\kappa$$

Forming ratios gives

$$\frac{v_y}{w_y} = 1 - \frac{v_x'v_x}{c^2} \tag{10}$$

and thus $v_y < w_y$. Viewed from XYZ, ball A has a greater y component of velocity than does ball B. (For ordinary velocities the difference is exceedingly small.)

Equation (10) requires the abandonment of one or the other of two principles of classical mechanics. If mass is an invariant, then the principle of conservation of linear momentum is violated in the y direction in XYZ. If the momentum principle is valid, then mass cannot be an invariant. The latter assumption is the one consistent with experiment, and will be the basis for what follows.

Let $m_A' = m_B'$ be the two masses in the $X'Y'Z'$ frame (they are equal by symmetry), and let $m_A \neq m_B$ be the two masses in the XYZ frame. Then $m_A w_y = m_B v_y$; thus,

$$\frac{m_B}{m_A} = \left(1 - \frac{v_x'v_x}{c^2}\right)^{-1}$$

This result can be rephrased entirely in terms of XYZ quantities by using (9) to substitute for v_x', which gives

$$\frac{m_B}{m_A} = \left(1 - \frac{v_x^2}{c^2}\right)^{-1/2} \tag{11}$$

This relation is seen not to depend on v_y and should hold even when $v_y = 0$. But then $w_y = 0$ as well, and—as seen from $X'Y'Z'$—the two balls approach each other along the X' axis and just barely touch as they pass. As seen from XYZ, ball A is at rest and ball B passes by, just touching A as it travels parallel to the X axis. With m_0 the mass of ball A at rest, (11) may be rewritten as

$$m_B = \frac{m_0}{(1 - v_x^2/c^2)^{1/2}}$$

One can now argue that it no longer matters whether ball A is present or not. Moreover, the rest mass of ball B should also be m_0, since in $X'Y'Z'$ one started with a symmetrical experiment using identical balls. With only ball B left, in constant rectilinear motion, the subscripts may be dropped on m_B and v_x, giving

$$m = \frac{m_0}{(1 - v^2/c^2)^{1/2}} \tag{12}$$

In (12), m_0 is the rest mass of ball B in XYZ, and m is its dynamic mass when going at a speed v relative to XYZ.

It is inferred from this result that the mass of any material body depends on its relative motion, increasing with speed according to (12). A clear confirmation has been given by Zahn and Spees.[7]

The transformation law for mass

Equation (12) is not, of course, the transformation law for mass because it yields the dynamic mass only in one frame of reference; but it can be used to relate dynamic mass in two different coordinate systems, as follows:

Let a body of rest mass m_0 have a velocity $\mathbf{v}(x, y, z, t)$ in

XYZ and a velocity $\mathbf{v}'(x', y', z', t')$ in $X'Y'Z'$. Then

$$m = \frac{m_0}{[1 - v^2/c^2]^{1/2}} \qquad m' = \frac{m_0}{[1 - (v')^2/c^2]^{1/2}}$$

are the expressions for the dynamic mass in the two coordinate systems. Thus

$$m' = m \left[\frac{1 - v^2/c^2}{1 - (v')^2/c^2}\right]^{1/2}$$

From the velocity transformation equations (9),

$$(v')^2 = (1 - uv_x/c^2)^{-2}[(v^2 - v_x^2)(1 - u^2c^2) + (v_x - u)^2]$$

so that

$$m' = \kappa(1 - uv_x/c^2)m \tag{13}$$

Equation (13) is the transformation law for mass. In using it one should remember that in general both m and m' are functions of time.

The transformation law for force

On the presumption that the Lorentz equations properly transform *all* the laws of physics (as required by the relativity principle) one may write

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \qquad \mathbf{F}' = \frac{d}{dt'}(m'\mathbf{v}')$$

and inquire what the force transformation law must be in order to derive either of these equations from the other through use of the Lorentz equations.

Using (9) and (13), the second of the preceding equations may be expanded to give

$$F_x' = \frac{dt}{dt'}\frac{d}{dt}[\kappa(v_x - u)m]$$

$$F_y' = \frac{dt}{dt'}\frac{d}{dt}[mv_y] \qquad F_z' = \frac{dt}{dt'}\frac{d}{dt}[mv_z] \tag{14}$$

From the differential expressions preceding (9),

$$\frac{dt}{dt'} = \frac{1}{\kappa(1 - uv_x/c^2)}$$

so that (14) becomes

$$F_x' = \frac{1}{1 - uv_x/c^2}\frac{d}{dt}(mv_x - mu) = \frac{F_x - u\, dm/dt}{1 - uv_x/c^2}$$

$$F_y' = \frac{1}{\kappa(1 - uv_x/c^2)}\frac{d}{dt}(mv_y) = \frac{F_y}{\kappa(1 - uv_x/c^2)}$$

$$F_z' = \frac{1}{\kappa(1 - uv_x/c^2)}\frac{d}{dt}(mv_z) = \frac{F_z}{\kappa(1 - uv_x/c^2)}$$

From (12), $dm/dt = \mathbf{v}\cdot\mathbf{F}/c^2$, so that finally

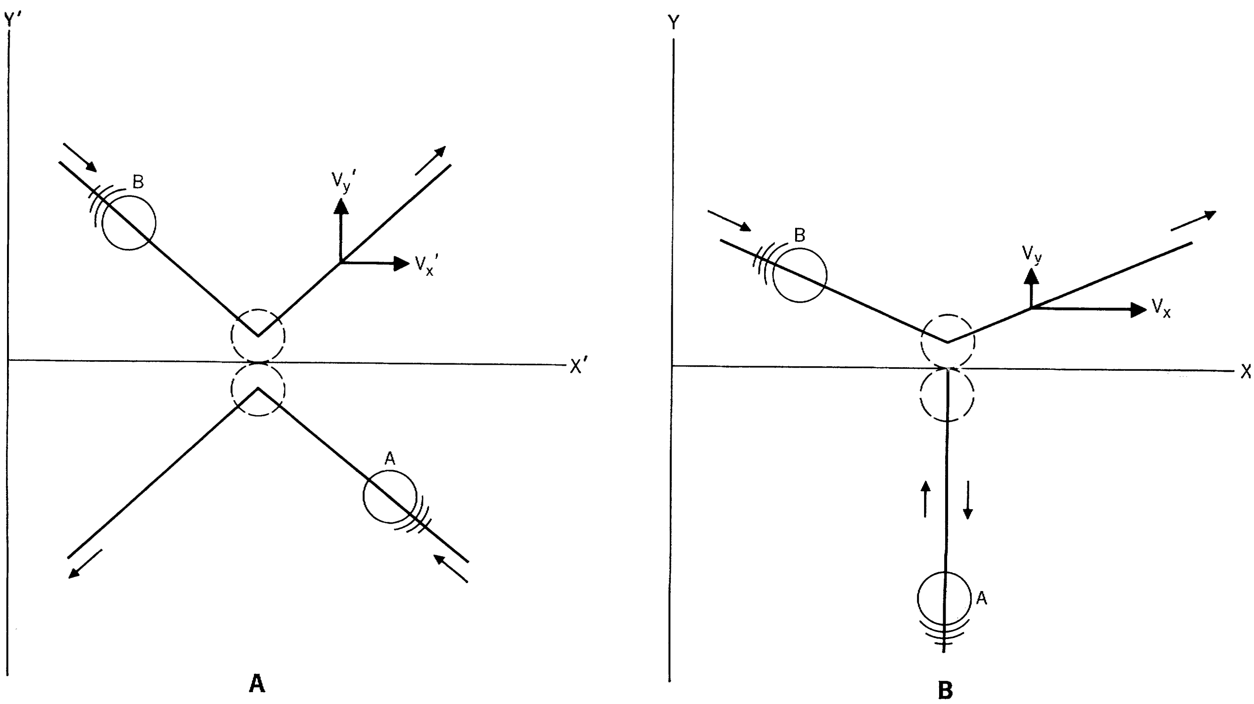
$$F_x' = F_x - \frac{uv_y/c^2}{1 - uv_x/c^2}F_y - \frac{uv_z/c^2}{1 - uv_x/c^2}F_z$$

$$F_y' = \frac{F_y}{\kappa(1 - uv_x/c^2)} \qquad F_z' = \frac{F_z}{\kappa(1 - uv_x/c^2)} \tag{15}$$

Equations (15) constitute the force transformation law. It is evident that if u and v are small compared to c, then $\mathbf{F}' \approx \mathbf{F}$, indicating that in such cases the classical expression, which equates these forces, is a valid approximation.

It is significant that the three equations in (15) are linear in the force components. Recalling that $\mathbf{F}$ or $\mathbf{F}'$ is the *total* force acting on the body of rest mass m_0, if $\mathbf{F}$ is composed of partial forces $\mathbf{F}_n$, then each of these partial forces has a counterpart, so that $\mathbf{F}'$ is composed of partial forces $\mathbf{F}_n'$. In general, the partial forces $\mathbf{F}_n$ are *independent*, and therefore the components of $\mathbf{F}_n'$ are also related to the components of $\mathbf{F}_n$ through (15). However, it should be recognized that when invoking (15) to relate partial forces, the *total* instantaneous velocity components of the mass must still be used.

Fig. 2. The collision of two balls.



Electrostatics

The only experimentally based postulate now needed to develop a complete electromagnetic theory is Coulomb's inverse-square law, which may be formulated as follows: Referring to Fig. 3, let there be a static assemblage of N charged particles, containing, respectively, charges $q_1, q_2, \ldots, q_n$ arbitrarily arranged in otherwise empty space. The quantities q_n are real numbers, which may be either positive or negative. The positions of these charged particles may be described in a coordinate system XYZ so that the nth particle is identified by the coordinates x_n, y_n, z_n or by the position vector $\mathbf{r}_n = \mathbf{1}_x x_n + \mathbf{1}_y y_n + \mathbf{1}_z z_n$.* These coordinates are not functions of time.

Additionally, let a particle containing a charge q be instantaneously at a point (x, y, z), described by the position vector $\mathbf{r} = \mathbf{1}_x x + \mathbf{1}_y y + \mathbf{1}_z z$. This charge will be permitted to move, so that the coordinates x, y, and z may be general functions of time. The total force exerted on q by the system of static charges q_n is then given by

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{N} \frac{qq_n}{\mathfrak{r}_n^2} \frac{\boldsymbol{\mathfrak{r}}_n}{\mathfrak{r}_n} = \frac{q}{4\pi\epsilon_0} \sum_{n=1}^{N} q_n \frac{\mathbf{r} - \mathbf{r}_n}{|\mathbf{r} - \mathbf{r}_n|^3} \tag{16}$$

in which

$$\boldsymbol{\mathfrak{r}}_n = \mathbf{r} - \mathbf{r}_n = \mathbf{1}_x(x - x_n) + \mathbf{1}_y(y - y_n) + \mathbf{1}_z(z - z_n)$$

so that $\boldsymbol{\mathfrak{r}}_n/\mathfrak{r}_n$ is a unit vector drawn from q_n toward q. (The symbol $\mathfrak{r}$ is a German lower-case x, chosen because of its resemblance to r. It may be called "r-cedilla.")

Equation (16) is a mathematical statement of Coulomb's law. The factor 4π is included in (16) so that it will not appear in the more often used Maxwell's equations. The factor ϵ_0 is called the permittivity of free space and is a units-adjusting parameter. When charge is measured in coulombs, force in newtons, and distance in meters, ϵ_0 has the measured value 8.854×10^{-12} farad per meter. These units form part of the MKS rationalized system, and will be used hereafter in this article.

Assuming that the charge q is small enough so that its presence or absence does not affect the spatial distribution of the charges q_n, the vector function

$$\mathbf{E} = \frac{\mathbf{F}}{q} = \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{N} q_n \frac{\boldsymbol{\mathfrak{r}}_n}{\mathfrak{r}_n^3} \tag{17}$$

is defined as the force per unit charge at (x, y, z). $\mathbf{E}$ may be expressed in units of newtons per coulomb and is variously called the electric force, the electric intensity, or the electric field strength.

By implication, if a charge q of *any* size is placed at (x, y, z), it experiences a force

$$\mathbf{F} = q\mathbf{E} \tag{18}$$

However, one must be careful in using (18) to ascertain that the presence of q has not disturbed the positions of the other charges. For example, if the assemblage of charges q_n is distributed over the surface of a conductor and a large charge q is placed in the vicinity, the charges q_n, being free to move, will redistribute themselves to new positions of equilibrium.

Equation (17) indicates that the electric force depends on the charges q_n and their positions relative to the point (x, y, z), but that it does *not* depend on q. An intensity

*Unit vectors will be designated by boldface Arabic numeral "one" (1) with a subscript.

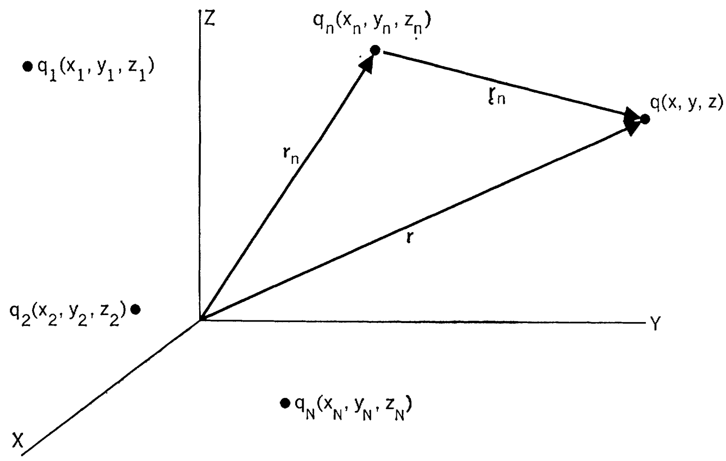


Fig. 3. Notation for Coulomb's law.

$\mathbf{E}(x, y, z)$ may thus be associated with the point (x, y, z) whether q is there or not. If the vector function $\mathbf{E}$ is interpreted in this manner, it may be taken as a fundamental subject of investigation. This is the field viewpoint of Faraday and Maxwell, which differs from the action-at-a-distance theories of their predecessors. In this view, the source charges q_n set up an electric field at the point (x, y, z); the field in turn will exert a force on any charge that might be introduced at (x, y, z). With this interpretation, E as defined by (17) is an *electrostatic* field, since the source points (x_n, y_n, z_n) are static, and the field point (x, y, z) has coordinates that are not connected to the possible motion of any particle.

In many problems it will be appropriate to consider the total charge $\rho\, dV$ in a volume element dV in lieu of the discrete charge q_n. In such cases, (17) may be written

$$\mathbf{E}(x, y, z) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\xi, \eta, \zeta) \frac{\boldsymbol{\mathfrak{r}}}{\mathfrak{r}^3}\, d\xi\, d\eta\, d\zeta \tag{19}$$

in which $\rho(\xi, \eta, \zeta)$ is the volume charge density function, expressed in coulombs per cubic meter, and $\boldsymbol{\mathfrak{r}}$ is drawn from the volume element centered at the source point (ξ, η, ζ) to the field point (x, y, z). The volume V is sufficient to encompass all the sources $\rho\, dV$.

Through the use of the three-dimensional Dirac delta function, an assemblage of *discrete* charges can be represented by a volume charge density function in the form

$$\rho(\xi, \eta, \zeta) = \sum_{n=1}^{N} q_n \delta(\mathbf{r}' - \mathbf{r}_n) \tag{20}$$

with $\mathbf{r}'$ the position vector drawn from the origin to (ξ, η, ζ). This representation can be verified by inserting (20) in (19), which yields (17).

Similarly, a surface charge distribution can be represented in terms of a volume charge density through the use of a one-dimensional delta function along the direction of the normal to the surface; a lineal charge distribution can be represented by a volume charge density through the use of a two-dimensional delta function in a transverse surface. By means of these representations, a general discussion in terms of ρ-type distributions has wide applicability, and (19) may be taken as the fundamental equation for the electrostatic field.

Use of the del operator

$$\nabla = \mathbf{1}_x \frac{\partial}{\partial x} + \mathbf{1}_y \frac{\partial}{\partial y} + \mathbf{1}_z \frac{\partial}{\partial z}$$

to form the gradient of inverse distance gives

$$\nabla \left(\frac{1}{\mathfrak{r}}\right) = -\frac{\boldsymbol{\mathfrak{r}}}{\mathfrak{r}^3}$$

from which it follows that (19) can be written

$$\mathbf{E}(x, y, z) = -\frac{1}{4\pi\epsilon_0} \int_V \rho(\xi, \eta, \zeta) \nabla \left(\frac{1}{\mathfrak{r}}\right) d\xi\, d\eta\, d\zeta$$

Since neither $\rho(\xi, \eta, \zeta)$ nor the limits of integration are functions of the field point (x, y, z), the order of integration and differentiation may be interchanged, yielding

$$\mathbf{E}(x, y, z) = -\nabla \int_V \frac{\rho(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta}{4\pi\epsilon_0 \mathfrak{r}} \tag{21}$$

Therefore the electric field is expressible as the negative of the gradient of the scalar function

$$\Phi(x, y, z) = \int_V \frac{\rho(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta}{4\pi\epsilon_0 \mathfrak{r}} \tag{22}$$

Φ is called the electrostatic potential function and is measured in volts. Since $\mathbf{E} = -\nabla\Phi$, the units of $\mathbf{E}$ are often given as volts per meter.

Use of the vector identity $\nabla \times \nabla f \equiv 0$ yields the information that

$$\nabla \times \mathbf{E} \equiv 0 \tag{23}$$

Formation of the divergence of (21) provides the companion expression

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \tag{24}$$

Since $\nabla \times \mathbf{E}$ and $\nabla \cdot \mathbf{E}$ jointly contain all the first partial derivatives of all three components of $\mathbf{E}$, if the curl and divergence of $\mathbf{E}$ are completely specified, as in (23) and (24), $\mathbf{E}$ itself can be uniquely determined.

Electromagnetics

The results just obtained, which link static electric fields to their static sources, may be enlarged to include time-varying sources and fields, by comparing the observations of two people in relative motion. To see this, imagine that an observer O' has created in free space a most general electrostatic field $\mathbf{E}'(x', y', z')$ through an arrangement of electric charges in the static distribution $\rho'(x', y', z')$. $\mathbf{E}'$ satisfies the equations

$$\nabla' \times \mathbf{E}' \equiv 0 \tag{25}$$

$$\nabla' \cdot \mathbf{E}' = \rho'/\epsilon_0 \tag{26}$$

Furthermore, observer O' will say that if a small test charge q' is instantaneously at some arbitrary point (x', y', z'), it will experience a force, due to the static charge system, given by

$$\mathbf{F}' = q'\mathbf{E}' \tag{27}$$

If the coordinate system $X'Y'Z'$ has its axes respectively aligned to those of an XYZ system, with the X' axis sliding along the $+X$ axis at a speed u, an observer O, who is stationary in XYZ, will see a moving system of sources and will deduce a force field that differs from the field observed by O'. The connection between these force fields may be determined through use of the force transformation law (15), which can be written alternatively as

$$\mathbf{F} = [\mathbf{1}_x F_x' + \kappa(\mathbf{1}_y F_y' + \mathbf{1}_z F_z')] + \kappa \frac{\mathbf{v}}{c} \times \left(\mathbf{1}_x \frac{u}{c} \times \mathbf{F}'\right) \tag{28}$$

in which $\mathbf{F}'$ is given by (27) and $\mathbf{v}(t)$ is the velocity of the test charge in XYZ. Two cases of (28) will be considered.

Case 1: $\mathbf{v} \equiv 0$

Here the test charge is at rest in XYZ and the force $\mathbf{F}$ is just the bracketed term in (28). Since the system charges are moving through XYZ at velocity $\mathbf{1}_x u$, the force $\mathbf{F}$ changes with time. If, for simplicity, charge is postulated as an invariant, observer O can define a time-varying electric field by the relation

$$\mathbf{F} = \mathbf{1}_x F_x' + \kappa(\mathbf{1}_y F_y' + \mathbf{1}_z F_z') = q\mathbf{E}(x, y, z, t) \tag{29}$$

Using (27), this gives the component equations

$$\begin{aligned} E_x(x, y, z, t) &= E_x'(x', y', z') \\ E_y(x, y, z, t) &= \kappa E_y'(x', y', z') \\ E_z(x, y, z, t) &= \kappa E_z'(x', y', z') \end{aligned} \tag{30}$$

Case 2: $\mathbf{v} \not\equiv 0$

Here the test charge q has an arbitrary motion $\mathbf{v}(t)$ in XYZ and the force $\mathbf{F}$ is the entire expression (28). Using the electric field defined by Case 1, together with (27), this can be written

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \left(\mathbf{1}_x \frac{\kappa u}{c^2} \times \mathbf{E}'\right) \tag{31}$$

The additional force, represented by the second term in (31), arises because of the motion of the test charge in XYZ. If an *additional field* $\mathbf{B}(x, y, z, t)$ is defined by

$$\mathbf{B} = \mathbf{1}_x \frac{\kappa u}{c^2} \times \mathbf{E}' \tag{32}$$

then (31) may be written

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{33}$$

Equation (33) is known as the Lorentz force law and is seen to be a relativistic transformation of the Coulomb force law (27).

The components of (32) are

$$\begin{aligned} B_x(x, y, z, t) &\equiv 0 \\ B_y(x, y, z, t) &= -\frac{\kappa u}{c^2} E_z'(x', y', z') \\ B_z(x, y, z, t) &= \frac{\kappa u}{c^2} E_y'(x', y', z') \end{aligned} \tag{34}$$

$\mathbf{B}(x, y, z, t)$ is known as the magnetic field and is measured in webers per square meter. (1 Wb/m² = 10^4 gauss.)

Equations (30) and (34) comprise the field transformation equations. They may be combined to give the inverse transformation

$$E_x' = E_x \qquad E_y' = \kappa(E_y - uB_z) \qquad E_z' = \kappa(E_z + uB_y) \tag{35}$$

The differential equations satisfied by the field $\mathbf{E}'$, namely (25) and (26), may be transformed with the aid of (35). However, it is convenient, as a preliminary step, to determine the transformation of the sources ρ'. Since charge has been taken as an invariant, $\rho\, dV = \rho'\, dV'$. But $dV' = \kappa\, dV$ because of length contraction in the X direction. Therefore,

$$\rho(x, y, z, t) = \kappa\rho'(x', y', z') \tag{36}$$

Since all of the source charges are moving through XYZ at a velocity $\mathbf{1}_x u$, they give rise to a *current* as seen by observer O. The distribution of this current may be deduced in the following way: At a general point (x, y, z), erect a volume element $dV = dx\, dy\, dz$, with $dx = u\, dt$. The charge enclosed at any time t is $\rho(x, y, z, t)dV$. All of this charge, and no other charge, will pass out of dV in time dt. The current flow is X-directed and given by

$$dI_x = \frac{\rho\, dV}{dt}$$

The *areal density* of current flow, in amperes per square meter, will therefore be

$$\iota_x = \frac{dI_x}{dy\, dz} = \frac{\rho u\, dt\, dy\, dz}{dt\, dy\, dz} = \rho u$$

Using (36), this becomes

$$\iota_x(x, y, z, t) = \kappa u\rho'(x', y', z') \tag{37}$$

Equations (36) and (37) constitute the source transformation equations. A *static* charge distribution in $X'Y'Z'$ is seen to transform into a time-varying charge distribution and a time-varying current density in XYZ.

With these results it is now possible to convert (25) and (26). The goal will be a set of equations in which the dependence of $\mathbf{E}$ and $\mathbf{B}$ on the sources is displayed.

To see how this is accomplished, consider any function f of the four coordinate variables. Upon making use of the Lorentz equations (6), one can establish that

$$\begin{aligned} \frac{\partial f}{\partial x'} &= \frac{\partial f}{\partial x}\frac{dx}{dx'} + \frac{\partial f}{\partial t}\frac{dt}{dx'} = \kappa\frac{\partial f}{\partial x} + \frac{\kappa u}{c^2}\frac{\partial f}{\partial t} \\ \frac{\partial f}{\partial t'} &= \frac{\partial f}{\partial t}\frac{dt}{dt'} + \frac{\partial f}{\partial x}\frac{dx}{dt'} = \kappa\frac{\partial f}{\partial t} + \kappa u\frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y'} &= \frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial z'} = \frac{\partial f}{\partial z} \end{aligned} \tag{38}$$

If f is not a function of t', then the second equation of (38) yields

$$\frac{\partial f}{\partial t} = -u\frac{\partial f}{\partial x} \tag{39}$$

Application of (38) and (39) to the curl of $\mathbf{E}'$ gives terms such as

$$\frac{\partial E_x'}{\partial z'} - \frac{\partial E_z'}{\partial x'} = \frac{\partial E_x'}{\partial z} - \kappa\frac{\partial E_z'}{\partial x} - \frac{\kappa u}{c^2}\frac{\partial E_z'}{\partial t}$$

which, with the use of (35), can be written

$$\frac{\partial E_x'}{\partial z'} - \frac{\partial E_z'}{\partial x'} = \frac{\partial E_x}{\partial z} - \kappa^2\left(\frac{\partial E_z}{\partial x} + u\frac{\partial B_y}{\partial x}\right) - \frac{\kappa^2 u}{c^2}\left(\frac{\partial E_z}{\partial t} + u\frac{\partial B_y}{\partial t}\right)$$

Upon determining all three components in this manner, one may write

$$\begin{aligned} \nabla' \times \mathbf{E}' \equiv 0 = \mathbf{1}_x &\left[\kappa\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \kappa u\nabla\cdot\mathbf{B}\right] + \\ \mathbf{1}_y &\left[\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + (1-\kappa^2)\frac{\partial E_z}{\partial x} - \kappa^2 u\frac{\partial B_y}{\partial x} - \frac{\kappa^2 u}{c^2}\frac{\partial E_z}{\partial t} - \frac{\kappa^2 u^2}{c^2}\frac{\partial B_y}{\partial t}\right] + \\ \mathbf{1}_z &\left[\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) - (1-\kappa^2)\frac{\partial E_y}{\partial x} - \kappa^2 u\frac{\partial B_z}{\partial x} + \frac{\kappa^2 u}{c^2}\frac{\partial E_y}{\partial t} - \frac{\kappa^2 u^2}{c^2}\frac{\partial B_z}{\partial t}\right] \end{aligned} \tag{40}$$

This result can be simplified considerably. From (30) and (34), $uE_z = -c^2B_y$ and $uE_y = c^2B_z$. When these relations are coupled with (39) and employed in (40), one finds that

$$\begin{aligned} 0 = \mathbf{1}_x &\left[\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + u\nabla\cdot\mathbf{B}\right] + \\ \mathbf{1}_y &\left[\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \frac{\partial B_y}{\partial t}\right] + \\ \mathbf{1}_z &\left[\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) + \frac{\partial B_z}{\partial t}\right] \end{aligned} \tag{41}$$

Further simplification is possible through determination of $\nabla\cdot\mathbf{B}$. Since

$$\begin{aligned} \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} &= \frac{\partial B_y}{\partial y'} + \frac{\partial B_z}{\partial z'} \\ &= -\frac{\kappa u}{c^2}\left(\frac{\partial E_z'}{\partial y'} - \frac{\partial E_y'}{\partial z'}\right) \end{aligned}$$

it follows from (25) that

$$\nabla\cdot\mathbf{B} \equiv 0 \tag{42}$$

and therefore that (41) reduces to

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \tag{43}$$

When this procedure is repeated for the divergence of $\mathbf{E}'$, one obtains

$$\begin{aligned} \nabla'\cdot\mathbf{E}' &= \frac{\rho'}{\epsilon_0} = \frac{\rho}{\kappa\epsilon_0} \\ &= \kappa\nabla\cdot\mathbf{E} - \kappa u\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \frac{\kappa u}{c^2}\frac{\partial E_x}{\partial t} \end{aligned} \tag{44}$$

Once again reduction is possible, since

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\partial E_x'}{\partial x} + \kappa\frac{\partial E_y'}{\partial y} + \kappa\frac{\partial E_z'}{\partial z} \\ &= \kappa\frac{\partial E_x'}{\partial x'} + \kappa\frac{\partial E_y'}{\partial y'} + \kappa\frac{\partial E_z'}{\partial z'} = \kappa\nabla'\cdot\mathbf{E}' \end{aligned}$$

Using (44), this gives

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \tag{45}$$

which reduces the remainder of (44) to the form

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \frac{\rho u}{c^2\epsilon_0} + \frac{1}{c^2}\frac{\partial E_x}{\partial t} \tag{46}$$

If a new constant μ_0, called the permeability of free space, is defined by the relation

$$\mu_0 = \frac{1}{c^2\epsilon_0} \tag{47}$$

then, with the substitution $\iota_x = \rho u$, (46) may be written

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \frac{\iota_x}{\mu_0^{-1}} + \frac{1}{c^2}\frac{\partial E_x}{\partial t} \tag{48}$$

Equations (42), (43), (45), and (48) comprise the transformation to XYZ space of the $X'Y'Z'$ field-source equations (25) and (26). They have been derived for the special case that $X'Y'Z'$ is in constant translative motion along the X axis. If a second frame $X''Y''Z''$, containing static sources, were similarly in motion parallel to the Y axis, the same procedure would yield four equations similar to these, the differences being that B_y would be zero and (48) would be replaced by its Y equivalent. Likewise, if a third frame $X'''Y'''Z'''$, containing static sources, were in constant motion parallel to the Z axis, this procedure would produce four equations distinguished by the characteristics that B_z would be zero and (48) would give way to its Z equivalent.

A linear superposition of the fields due to static sources in all other Lorentzian frames therefore yields *total* fields $\mathbf{E}(x, y, z, t)$ and $\mathbf{B}(x, y, z, t)$ in XYZ, which satisfy

$$\begin{aligned} \nabla \times \mathbf{E} &= -\dot{\mathbf{B}} \qquad \nabla \times \mathbf{B} = \frac{\boldsymbol{\iota}}{\mu_0^{-1}} + \frac{\dot{\mathbf{E}}}{c^2} \\ \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \mathbf{B} \equiv 0 \end{aligned} \tag{49}$$

These four equations are known as Maxwell's equations, and have been derived here only for sources in free space. Upon representing materials by equivalent sources, (49) is easily extended to apply to general media.

It is evident that observer O need not rely on the static sources of O', O'', etc., to establish his time-varying electromagnetic fields, but can do this equally well himself by direct creation of the time-varying sources ρ and $\boldsymbol{\iota}$. That most general sources ρ, $\boldsymbol{\iota}$ may be treated as a superposition of *static* sources in $X'Y'Z'$, $X''Y''Z''$, etc., is demonstrated in the Appendix.

Conclusions

Through use of the relativisitic force transformation equations, one is able to show that the Lorentz force law is a transformation of the electrostatic Coulomb force equation. The time-varying electric and magnetic fields, which are defined as constituent parts of the Lorentz force expression, are then found to satisfy Maxwell's equations. This establishes the identity of the fields appearing in Maxwell's equations and the Lorentz force law. Additionally, Maxwell's equations are seen to be transformations of the expressions for the curl and divergence of an electrostatic field. All of the conventional relations of electrostatics, magnetostatics, and electromagnetics are derivable from Maxwell's equations. Therefore, this procedure leads to a relativistically exact, complete electromagnetic theory, using Coulomb's inverse-square law as the sole experimentally based electrical postulate. Although no new relations are uncovered, this approach has the virtue of enriching one's understanding of the subject by revealing the fundamental unity of electric and magnetic phenomena.

Appendix

Let a general current-density distribution $\boldsymbol{\iota}(x, y, z, t)$ be represented by the fourfold Fourier integral

$$\boldsymbol{\iota}(x, y, z, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{g}(k_x, k_y, k_z, \omega)\cdot e^{j(\omega t + k_x x + k_y y + k_z z)}\, dk_x\, dk_y\, dk_z\, d\omega \tag{A.1}$$

In a similar manner, let a general charge-density distribution $\rho(x, y, z, t)$ be represented by

$$\rho(x, y, z, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_x, k_y, k_z, \omega)\cdot e^{j(\omega t + k_x x + k_y y + k_z z)}\, dk_x\, dk_y\, dk_z\, d\omega \tag{A.2}$$

The integrands of (A.1) and (A.2) are connected by the continuity equation, $\nabla \cdot \boldsymbol{\iota} = -\dot{\rho}$, which gives

$$f = -\frac{1}{\omega}(\mathbf{k} \cdot \mathbf{g}) \tag{A.3}$$

where $\mathbf{k} = \mathbf{1}_x k_x + \mathbf{1}_y k_y + \mathbf{1}_z k_z$.

If the fictitious charge and current densities in the interval $(d\mathbf{k}, d\omega)$ are treated as an independent entity that satisfies the flow equation $\boldsymbol{\iota} = \rho\mathbf{v}$, then the velocity of these fictitious charges is

$$\mathbf{v}(\mathbf{k}, \omega) = \frac{\mathbf{g}}{f} = -\frac{\omega\mathbf{g}}{\mathbf{k}\cdot\mathbf{g}} \tag{A.4}$$

This velocity is *independent* of x, y, z, and t, and is therefore a common velocity shared by all the charges that give rise to the $(\mathbf{k}, \omega)$ current and charge waves. In a coordinate system traveling at the velocity $\mathbf{v}$ with respect to XYZ, these charges are at rest. As $\mathbf{k}$ and ω are permitted to range over their complete spectra of values, (A.4) indicates that all values of $\mathbf{v}$ will be encountered in the interval $0 \leq v < \infty$. One may conclude from this that arbitrary *static* charge distributions in all Lorentzian frames may be combined to give the most general time-varying spatial distributions of current and charge density in a particular Lorentzian frame.

Because the range of $\mathbf{v}$ is unrestricted, some of these fictitious charge distributions are traveling through XYZ at speeds greater than light. This requires use of the Lorentz transformation equations when $v > c$. Even though the transformation is then nonphysical, this is mathematically admissible in the sense that Maxwell's equations transform properly under a Lorentz transformation, regardless of the value of v/c. It should be emphasized that the charge densities in the interval $(d\mathbf{k}, d\omega)$ are *fictitious*. No intimation is intended that the *real* time-varying charges, which are the sum of these fictitious static charge densities, are traveling at speeds in excess of c.

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