circuit was designed with a flat 10-dB gain in the frequency band Ω_0. Then beginning with these initial parameter values, suppose it is desired to obtain a flat gain at 12 dB. The Rosenbrock algorithm usually would not converge. It appeared that a strong local minimum existed at 10 dB. However, convergence could be obtained by including the reflection loss terms in the performance index, or by starting with new initial parameter values for which the gain curve was not very flat. In the case of the antenna load, convergence to a reasonable solution was obtained only by minimizing the reflection loss terms. Each design took approximately two minutes of 360/75 time except for the two-stage amplifier which required approximately five minutes.

Additional problems being studied are the design of broad-band low-noise receivers and the design of distributed filters. Also, other minimization algorithms which use the gradient of E are being studied at the present time.

ACKNOWLEDGMENT

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Wave Propagation in Hollow Conducting Elliptical Waveguides

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Abstract—**The propagation of electromagnetic waves in a hollow perfectly conducting pipe with an elliptical cross section and the results of numerical calculations of the cutoff wavelength of nineteen successive modes are presented. Some inaccuracies in the usual mode classification are proven and corrected. As a large number of numerical calculations are required to determine the cutoff wavelength for a single set of dimensions and a single mode, approximate formulas for the eight lowest order modes are suggested. These formulas are of a simple algebraic form and give a relative error smaller than 0.25 percent. With the exact succession of the different modes it becomes possible to compare the bandwidth of an elliptical waveguide to the bandwidth of the rectangular and circular guide. The measured values of the cutoff wavelength of different modes agree very well with the theoretical calculated values.**

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I. INTRODUCTION

IN 1938 Chu [2] presented the theory of the transmission of electromagnetic waves in hollow conducting pipes of elliptic cross section. He obtained numerical results for the cutoff frequency and the attenuation for six different waves. In 1947 Kinzer and Wilson [3] published the first approximate formulas to determine the cutoff frequency of the TE_{c01}, TM_{c11}, and TM_{s11} modes for a given elliptical cross section. The authors did not mention the degree of accuracy of their formulas. The impedance of an elliptic waveguide operating in the TE_{c11} mode was discussed by Valenzuela in a paper published in 1960 [5]. Using new approximations for the modified Mathieu functions, Piefke [7] obtained attenuation constants for twelve modes. The optimum design dimensions for minimum attenuation, when an elliptical waveguide is used in the fundamental mode (TE_{c11}) and the practical characteristics of a long single-piece flexible aluminum waveguide of approximately elliptical cross section were presented by Maeda [9], [10] in 1968.

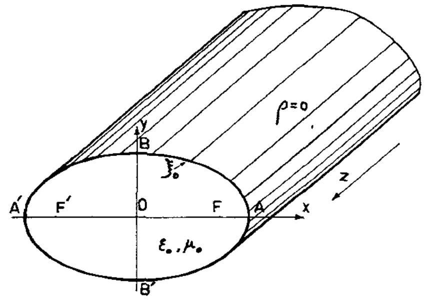
The mean advantages of elliptical waveguides are that long continuous lines are easily manufactured and transported. Furthermore, there is no mode splitting or rotation of the polarization plane for slight deformations of the cross section while simple matched connecting parts to rectangular and circular waveguides are possible. Although elliptical waveguides are commercially available and have already been used in several applications such as multichannel communication and radar feed lines, there still remain a lot of unresolved theoretical and practical problems in this domain.

This paper presents the results of exact numerical calculations of the cutoff frequency of nineteen modes in an elliptic waveguide. The calculations were made on an IBM 360/44 using Bessel function product series for the modified Mathieu functions of the first kind [1], [14], [15].

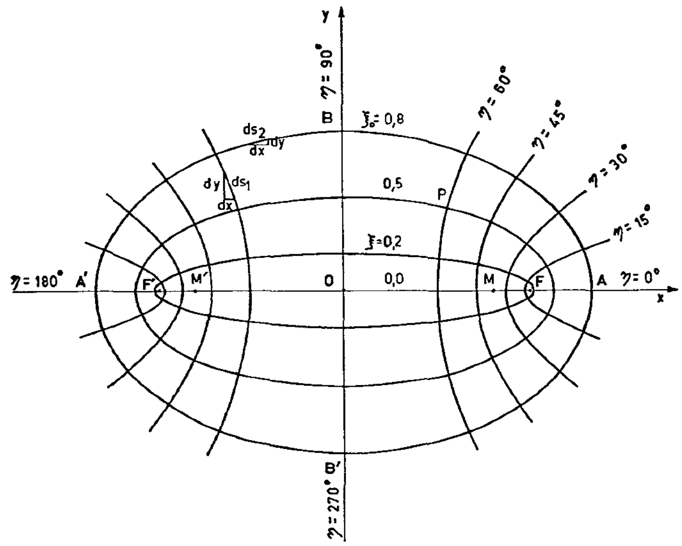
The exact succession of the modes permits a comparison of the bandwidth of elliptical waveguides with the bandwidth of rectangular and circular waveguides. For eccentricities between 0.05 and 0.95, practical formulas for the determination of the cutoff wavelength are presented for the eight lowest order modes. They are quite simple and give a relative error smaller than 0.25 percent. The theoretical results are compared with measurements on a waveguide operating in the 2–4-GHz frequency band. The difference between calculated and measured values is smaller than 0.6 percent.

II. Wave Equation for a Perfect Elliptical Waveguide

We assume an air-filled hollow-piped uniform waveguide of elliptical cross section with a perfect conducting wall [Fig. 1(a)]. Orthogonal elliptic coordinates (ξ, η, z) are used to solve Maxwell's equations. Such a coordinate system is shown on Fig. 1(b). The contour surfaces of constant ξ are confocal elliptical cylinders while those of constant η are confocal hyperbolic cylinders. The distance between the foci F, F' is $2h$. The confocal cylinder $\xi=\xi_0$ forms the inner boundary of the waveguide while the z axis coincides with the longitudinal axis of the pipe. The eccentricity e of the cross section is given by $1/\cosh \xi_0$, while the major axis (AA') and the minor axis (BB') of the ellipse are $2a=2h \cosh \xi_0$ and $2b=2h \sinh \xi_0$. A harmonic time variation and a propagation in the positive z direction are assumed. In complex representation these assumptions result in a multiplication of all wave functions by $\exp(j(\omega t-\beta z))$. For a given mode the phase factor β is a function of the cross section and the frequency of the wave and is to be determined from the boundary conditions. As the investigated elliptical waveguide is a homogeneous, simple, and perfect guide [12], all modes will have either $H_z\equiv 0$ (TM modes) or $E_z\equiv 0$ (TE modes).



(a)



(b)

Fig. 1. (a) Elliptical waveguide. (b) Orthogonal elliptical coordinate system.

From Maxwell's equations we get the following wave equation for E_z(TM) or H_z(TE):

$$\left[\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}+2q\,(\cosh 2\xi-\cos 2\eta)\right]\begin{bmatrix}E_z\\ H_z\end{bmatrix}=0 \qquad (1)$$

where

$$4q = k_c^2 h^2 \qquad (2)$$

$$k_c^2 = \omega^2\epsilon_0\mu_0-\beta^2 = 4\pi^2/\lambda_c^2 \qquad (3)$$

$$h = ae. \qquad (4)$$

Using the method of separation of variables we obtain the following solution for the wave equation:

$$\begin{bmatrix}E_z\\ H_z\end{bmatrix}=\begin{bmatrix}C_m Ce_m(\xi, q)ce_m(\eta, q)\\ S_m Se_m(\xi, q)se_m(\eta, q)\end{bmatrix}\exp\,(j(\omega t-\beta z)). \qquad (5)$$

In these equations $ce_m(\eta, q)$ and $se_m(\eta, q)$ are ordinary even and odd Mathieu functions while $Ce_m(\xi, q)$ and $Se_m(\xi, q)$ are the corresponding modified Mathieu functions of the first kind and order m. The other field components are easily obtained by applying Maxwell's equations. As may be seen from (5) there are four types of propagation in an elliptical waveguide, namely, even and odd TE and TM modes. To distinguish between the different modes the first index of each mode designation will be c (cos-type) for an even mode and s (sin type) for an odd mode, while the second index m is related to the order of the Mathieu function. Furthermore, the following equations must hold to satisfy the boundary conditions on the wall:

$$\text{TM modes:}\quad \begin{aligned} Ce_m(\xi_0, q) &= 0 \quad \text{(even)} \\ Se_m(\xi_0, q) &= 0 \quad \text{(odd)} \end{aligned} \tag{6}$$

$$\text{TE modes:}\quad \begin{aligned} Ce_m'(\xi_0, q) &= 0 \quad \text{(even)} \\ Se_m'(\xi_0, q) &= 0 \quad \text{(odd)} \end{aligned} \tag{7}$$

with

$$\cosh \xi_0 = e^{-1}. \tag{8}$$

As the parameter q is related to the cutoff wave number k_c by (2), we get a different mode for each root of (6) and (7). To resolve this ambiguity, a third subscript n, corresponding to the nth parametric root, is required in the mode designation. With (2) the general formula for the cutoff wavelength of a TE or TM mode in an elliptical waveguide becomes

$$\lambda_c = \frac{\pi a e}{\sqrt{q}} \cdot \tag{9}$$

For a $\mathrm{TM}_{cmn}(\mathrm{TM}_{smn})$ mode $q = q_{cmn}(q_{smn})$ is the nth parametric zero of the even (odd) modified Mathieu function of order m with argument ξ_0. For a $\mathrm{TE}_{cmn}(\mathrm{TE}_{smn})$ mode $q = \bar{q}_{cmn}(\bar{q}_{smn})$ is the nth parametric zero of the first derivative of the same function.

III. The Cutoff Wavelength as a Function of the Eccentricity and Mode

According to (9) the cutoff wavelength of a given mode is a function of the geometry of the cross section and the parameter q. This parameter being also a function of the mode and eccentricity (6) and (7), it follows from (9) that the cutoff wavelength of any mode is completely determined by the dimensions of the elliptical cross section. This relation may be represented in different ways.

A. The Function $q = f(e)$

It is clear that the exact computation of the modified Mathieu functions forms the main difficulty in the study of elliptical waveguides. These functions may be calculated by means of hyperbolic functions series, Bessel functions series, and Bessel functions product series. It is proved that these series and their pth derivatives are absolutely and uniformly convergent in any finite region of the ξ plane or in any closed interval of ξ real [1].

The Bessel functions product series are most suitable for practical computer calculations as they have the highest rate of convergence. In a recent paper [14], [15] it was shown that the rate of convergence of the series with hyperbolic functions is so slow that they are only useful for the lowest order functions and for a limited interval of the argument ξ and the parameter q.

As the computations for the different modes are similar we shall only deal with the TM_{c11} mode as an example. The three possible series for the involved modified Mathieu function $Ce_1(\xi, q)$ are

$$Ce_1(\xi, q) = \sum_{j=0}^{\infty} A_{2j+1} \cosh(2j+1)\xi \tag{10}$$

$$Ce_1(\xi, q) = \sum_{j=0}^{\infty} D_{2j+1} J_j(2\sqrt{q}\cosh \xi) \tag{11}$$

$$\begin{aligned} Ce_1(\xi, q) = \sum_{j=0}^{\infty} C_{2j+1}[&J_j(\sqrt{q}e^{-\xi})J_{j+1}(\sqrt{q}e^{\xi}) \\ &+ J_j(\sqrt{q}e^{\xi})J_{j+1}(\sqrt{q}e^{-\xi})] \end{aligned} \tag{12}$$

with

$$D_{2j+1} = (-1)^{j+1} \frac{ce_1'(\pi/2, q)}{\sqrt{q}\, A_1} A_{2j+1} \tag{13}$$

$$C_{2j+1} = (-1)^{j+1} \frac{ce_1(0, q) ce_1'(\pi/2, q)}{\sqrt{q}\, |A_1|^2} A_{2j+1} \tag{14}$$

$$ce_1(0, q) = \sum_{j=0}^{\infty} A_{2j+1} \tag{15}$$

$$ce_1'(\pi/2, q) = \sum_{j=0}^{\infty} (-1)^{j+1}(2j+1) A_{2j+1} \tag{16}$$

and with J_j, J_{j+1} Bessel functions of the first kind.

The function $y = Ce_1(\xi, q)$ is a solution of the modified Mathieu equation

$$y'' - (a - 2q \cosh 2\xi) y = 0 \tag{17}$$

if a equals the characteristic number a_1, which is the first root of the following infinite continued fraction

$$V_1 - 1 - \frac{1}{V_{3} -} \frac{1}{V_{5} -} \cdots \frac{1}{V_{2j+1} -} \cdots = 0$$

where

$$V_{2j+1} = \frac{a - (2j+1)^2}{q} \quad (j \geq 0).$$

Once a_1 has been determined, the coefficients A_{2j+1} are obtained using the following relations:

$$\begin{aligned} A_1 &= 1 \\ A_{2j+1} &= G_{2j+1} A_{2j-1} \quad (j > 0) \\ G_{2j+1} &= \frac{1}{V_{2j+1} -} \frac{1}{V_{2j+3} -} \cdots \quad (j > 0). \end{aligned}$$

Starting with A_3, we compute the successive coefficients until we reach one smaller in absolute value than a given limit ϵ. The infinite series are then truncated and limited to that last term, say A_{2n+1}, D_{2n+1}, and C_{2n+1}. Defining

$$S = \sum_{j=0}^{n} |A_{2j+1}|^2$$

we normalize the coefficients by dividing them by the square root of S or

$$A_{2j+1} = \frac{A_{2j+1}}{\sqrt{S}} \cdot$$

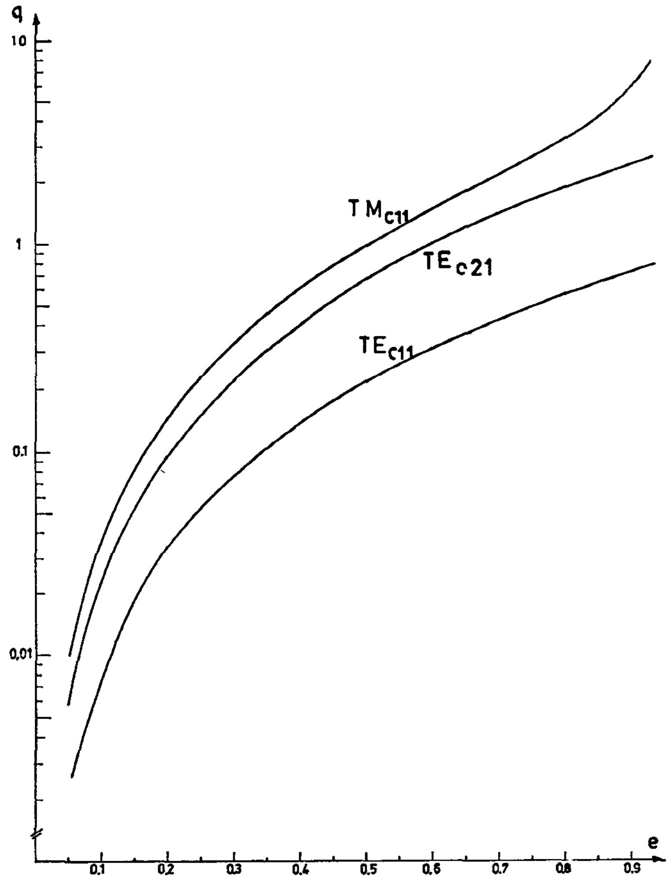


Fig. 2. Function $q=f(e)$ for TM_{c11}, TE_{c11}, and TE_{c21} mode.

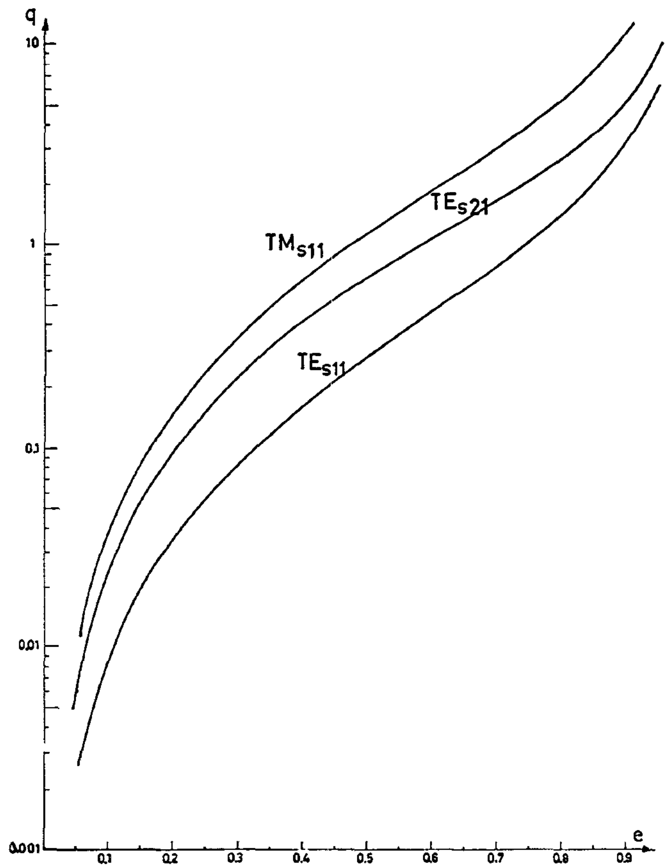


Fig. 3. Function $q=f(e)$ for TM_{s11}, TE_{s11}, and TE_{s21} mode.

The corresponding coefficients D_{2j+1} and C_{2j+1}, which are functions of A_{2j+1} and q alone, are easily obtained with (13) and (14). As it is much easier to determine the zeros than the parametric zeros, the first root ξ_0 of $Ce_1(\xi, q_{c11})=0$ is evaluated for a given q_{c11} value. With (8) we obtain then the required eccentricity of the cross section. Figs. 2–4 give the parameter q as a function of the eccentricity for the eight lowest order waves and of eccentricities between 0.05 and 0.95.

B. The Function $\lambda_c/P=h(e)$

The perimeter P of an ellipse with eccentricity e and major axis $2a$ is given by

$$P = 4aE(e^2) \tag{18}$$

with $E(e^2)$ the complete elliptic integral of the second kind. With (9) and (18) the ratio of the cutoff wavelength to the perimeter becomes

$$\frac{\lambda_c}{P} = \frac{\pi e}{4\sqrt{q}E(e^2)}\,. \tag{19}$$

From (19) we conclude that the ratio of the cutoff wavelength of a given mode to the perimeter of the cross section is a function of the eccentricity alone.

The results for the eight lowest order modes are given in Fig. 5. This method of representation is classic and

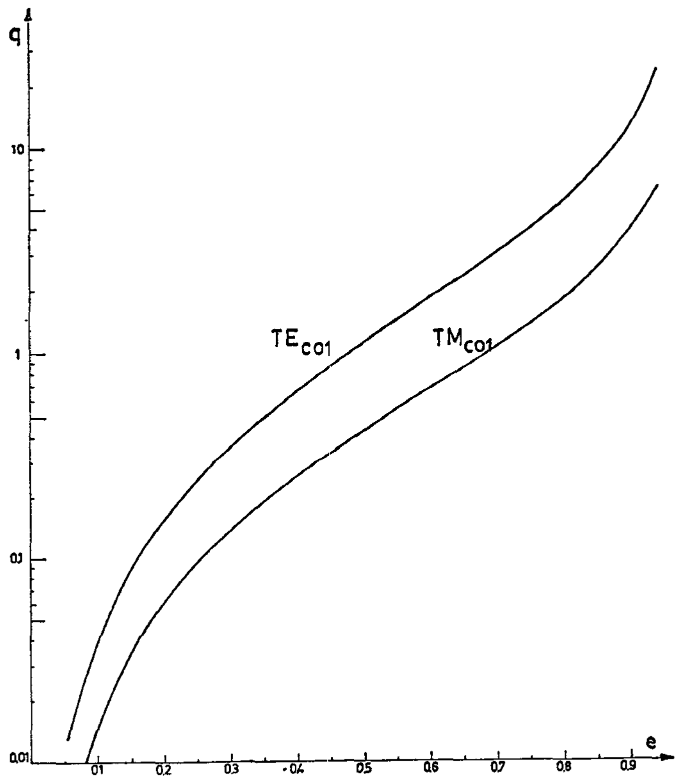


Fig. 4. Function $q=f(e)$ for TE_{c01} and TM_{c01} mode.

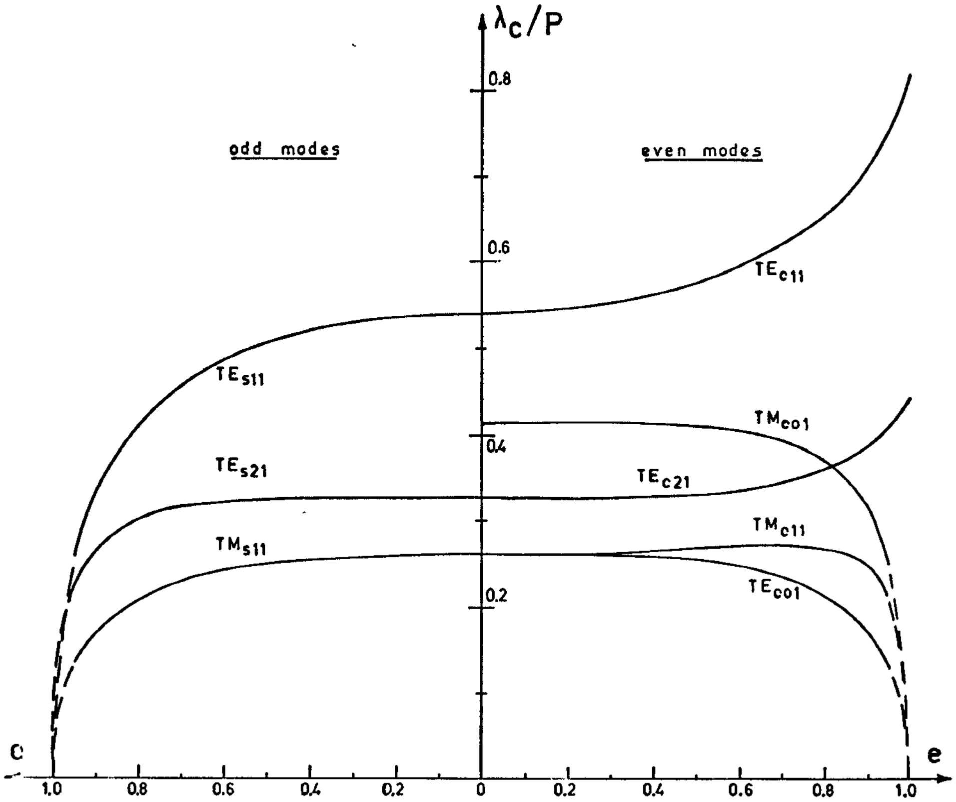


Fig. 5. Function $\lambda_c/P = h(e)$ for eight lowest order waves.

was introduced by Chu [2]. Using Bessel functions series, he obtained exact results for the TE_{c11}, TE_{s11}, TM_{c01}, TM_{c11}, TE_{c01}, and TM_{s11} modes and this for eccentricities between 0.0 and approximately 0.8. From Fig. 5 it is clear that these six modes given in most publications and textbooks are not the lowest order waves, as the even and odd TE_{21} modes were forgotten in the classification. In 1964 Piefke derived new asymptotic formulas for the modified Mathieu functions [8]. By means of these formulas and the Bessel functions series he extended the λ_c/P chart with the TM_{c02}, TE_{c12}, TE_{s12}, TM_{c12}, TE_{c02}, and TM_{s12} modes [6], [7]. It is clear that the even and odd TE_{21} modes were omitted again. Furthermore, there are several other higher order modes with a cutoff wavelength greater than the cutoff wavelength of the modes given in [7].

As to the asymptotic approximations for the modified Mathieu functions we must remark that they have a sufficient accuracy only if $2\sqrt{q}\cosh\xi$ is much greater than unity. For the determination of the cutoff wavelength of an elliptical waveguide, this condition becomes

$$2\sqrt{q}\cosh\xi_0 = 2\sqrt{q}e^{-1} \gg 1$$

or with (9)

$$\lambda_c/a \ll 2\pi. \tag{20}$$

The consequences of this condition are discussed in Section III-C.

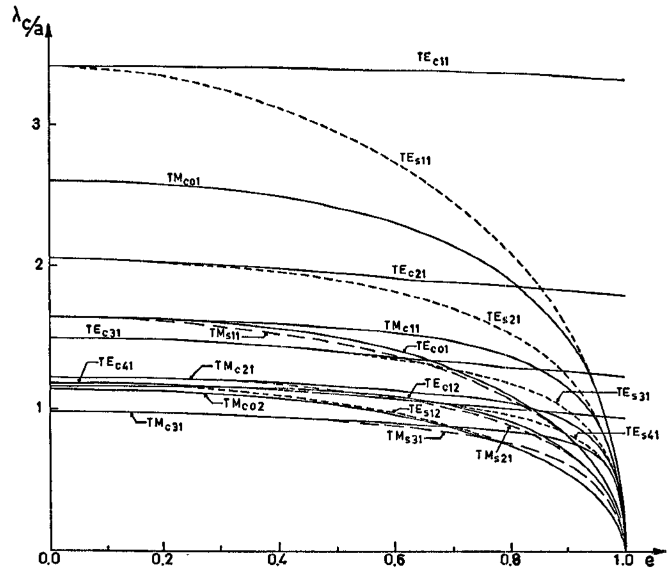


Fig. 6. Function $\lambda_c/a = g(e)$ for nineteen successive modes.

C. *Function $\lambda_c/a = g(e)$*

The general formula (9) may be written as

$$\frac{\lambda_c}{a} = \frac{\pi e}{\sqrt{q}} = g(e). \tag{21}$$

This function was computed for nineteen successive modes and for eccentricities between 0.0 and 0.95 at least. The results are represented on Fig. 6. This figure

TABLE I

Mode	Formula	Interval of e	$\lvert\epsilon_r\rvert_{\max}$ (percent)
TE_{c11}	$\bar{q}_{c11}=0.8476e^2-0.0013e^3+0.0379e^4$	[0.0, 0.4]	0.01
	$\bar{q}_{c11}=-0.0064e+0.8838e^2-0.0696e^3+0.0820e^4$	[0.4, 1.0]	0.02
TE_{c21}	$\bar{q}_{c21}=0.0001e+2.3260e^2+0.0655e^3+0.9816e^4$	[0.0, 0.42]	0.07
	$\bar{q}_{c21}=-0.0060e+2.1493e^2+0.9476e^3-0.0532e^4$	[0.42, 1.0]	0.07
TE_{c01}	$\bar{q}_{c01}=-0.0073e+3.8569e^2-1.3105e^3+4.6802e^4$	[0.05, 0.45]	0.3
	$\bar{q}_{c01}=-1.2264-1.3936e+1.5515e^2+1.3156/(1-e)$	[0.45, 0.95]	0.3
TE_{s11}	$\bar{q}_{s11}=-0.0018e+0.8974e^2-0.3679e^3+1.612e^4$	[0.05, 0.50]	0.4
	$\bar{q}_{s11}=-0.1483-1.0821e+1.0829e^2+0.3493/(1-e)$	[0.50, 0.95]	0.5
TE_{s21}	$\bar{q}_{s21}=-0.0053e+2.4700e^2-0.9098e^3+2.8655e^4$	[0.05, 0.60]	0.5
	$\bar{q}_{s21}=1.0692-5.2863e+5.9122e^2+0.4171/(1-e)$	[0.60, 0.95]	0.5
TM_{c01}	$q_{c01}=-0.0016e+1.488e^2-0.314e^3+1.425e^4$	[0.05, 0.50]	0.2
	$q_{c01}=-0.222-0.728e+1.308e^2+0.341/(1-e)$	[0.50, 0.95]	0.2
TM_{c11}	$q_{c11}=-0.0049e+3.7888e^2-0.7228e^3+2.2314e^4$	[0.05, 0.55]	0.3
	$q_{c11}=-0.1379-1.3138e+3.9307e^2+0.4056/(1-e)$	[0.55, 0.95]	0.3
TM_{s11}	$q_{s11}=-0.0063e+3.8316e^2-1.1351e^3+5.2229e^4$	[0.05, 0.45]	0.3
	$q_{s11}=-1.2014-1.6271e+2.1684e^2+1.3089/(1-e)$	[0.45, 0.95]	0.3

illustrates very well the transition from elliptical to circular cross section and is easier to use than Fig. 5 as the latter requires the computation of the perimeter of the elliptical cross section. The classification of the first nineteen modes clearly shows that there exist several other modes between the TM_{c02} and TM_{c12} modes. This point was not made before. Furthermore, it turns out that the cutoff wavelength of the TE_{c02} and the even and odd TM_{12} modes is not only smaller than the TM_{c31} mode but also smaller than the even and odd TE_{51} and TE_{22} modes which are not represented on Fig. 6. It is also obvious that the succession of the various modes is a function of the eccentricity and that for $e=0$ this succession becomes those of a circular waveguide. The cutoff wavelength of the TE_{cm1} mode ($m=1, 2, \cdots$) does not vary very fast with the eccentricity and reaches a well-defined nonzero value for e equal to unity. For all other modes the cutoff wavelength tends to zero as the eccentricity tends to unity.

Another interesting phenomenon is the fact that the difference between an even and odd mode of the same type and order becomes less pronounced for higher order waves. As to condition (20) we notice that the asymptotic formulas are useful for great values of the eccentricity or for very high-order waves. Nevertheless, we must remark that condition (20) is more severe for even than for odd modes and also becomes more severe accordingly as the order of the modified Mathieu function becomes greater.

IV. Approximate Formulas for the Function $q=f(e)$

The determination of the exact value of the parameter q for a given mode and eccentricity is rather complicated as may be seen from the preceding discussion. To avoid this complexity, we derived an approximate analytic

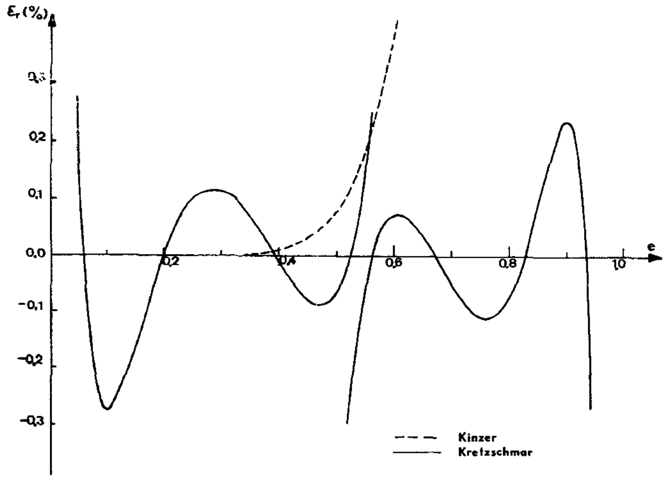


Fig. 7. Relative-error curve of approximate formula for function $q_{c11}=f(e)$.

relation between e and q for the eight lowest order modes. For practical use, it is not only necessary to ensure a known and sufficient accuracy but also to have a simple algebraic form.

Approximate formulas for the TE_{c01}, TM_{c11}, and TM_{s11} modes have been proposed by Kinzer and Wilson [3]. These formulas are approximations for the function $\lambda_c/a=g(e)$ and have the following general form:

$$\frac{\lambda_c}{a}=\frac{E(e^2)}{a_0+a_1E+a_2E^2+a_3E^3} \tag{22}$$

where $E=1-\sqrt{1-e^2}$ and $E(e^2)$ is the complete elliptic integral of the second kind.

The accuracy of the proposed formulas is very good for small eccentricities but drops for larger values

(e greater than 0.6 or 0.75, depending upon the mode). Dividing the involved interval of eccentricity (0.05 to 0.95) in two parts and adopting the maximum error criterion [17], it is possible to approximate the function $q=f(e)$ by $q=a_1e+a_2e^2+a_3e^3+a_4e^4$ for small eccentricities and by $q=b_0+b_1e+b_2e^2+b_3/(1-e)$ for larger values. The coefficients a_i, b_j are determined by means of a set of four exact function values with arguments corresponding to the four zeros of the Chebyshev polynomial $T_4(e)$ in the involved interval of e. The optimum intervals are determined experimentally.

The formulas for the different modes are given in Table I. The range of validity and the absolute value of the maximum relative error is given for each formula. The relative-error curve of the approximation for the TM_{c11} mode is given as an example on Fig. 7. On the same figure the relative-error curve of the formula given by Kinzer and Wilson is drawn. The formulas for the other modes have a similar error curve. When the formulas of Table I are used to determine the cutoff wavelength, the maximum relative error is only one half of the value given in the last column of Table I.

V. Bandwidth of Elliptical Waveguides

As to the bandwidth of elliptical waveguides, it is clearly shown by Fig. 6 that the largest possible bandwidth is obtained for eccentricities greater than approximately 0.86. This corresponds to an axial ratio of 0.5. The bandwidth of an elliptical waveguide with axes $2a$ and $2b$ is compared with the bandwidth of a rectangular waveguide with dimensions $2a$ and $2b$ and with a circular waveguide with radius a.

The results for the various types are given in Table II for $a=2b$, i.e., $e=0.866$.

From Table II it follows that the cutoff wavelength of the dominant TE_{c11} mode in the elliptical waveguide is smaller than the cutoff wavelength of the dominant mode in the circular and rectangular waveguide. The first higher order mode in the elliptical waveguide appears at a higher frequency than in the rectangular or circular waveguide. The bandwidth of an elliptical guide with $a=2b$ is 25 percent less than the bandwidth of the corresponding rectangular waveguide but nearly the double of the circular waveguide.

TABLE II

Type	Dominant Mode	λ_c	First Higher Order Mode	λ_c	Bandwidth
Rectangular	TE_{10}	$4.00\,a$	TE_{01} TE_{20}	$2.00\,a$	$2.00\,a$
Circular	TE_{11}	$3.41\,a$	TM_{01}	$2.61\,a$	$0.80\,a$
Elliptical	TE_{c11}	$3.35\,a$	TE_{c21}	$1.84\,a$	$1.51\,a$

VI. Experimental Results

To check the validity of the calculations, some experiments were carried out on an elliptical resonant cavity with the following dimensions: major axis $2a=21.55$ cm, eccentricity $e=0.66$, and length $L=29.865$ cm. The cutoff wavelength of a given mode is calculated from the resonant frequencies of the cavity. The results for the TM_{c11} mode are given here as an example. Table III gives the measured values of λ_{c11} determined from the successive resonant frequencies f_0 of the cavity. The exact value of λ_{c11} for the given waveguide is 16.21 cm. As may be seen, the relative error on λ_{c11} is always smaller than 0.6 percent. The results for the other modes are similar.

TABLE III

Mode	f_0 (MHz)	λ_{c11}(cm)
TM_{c112}	2111	16.16
TM_{c113}	2389	16.18
TM_{c114}	2727	16.26
TM_{c115}	3126	16.12
TM_{c116}	3534	16.25
TM_{c117}	3978	16.12

VII. Conclusions

Using Bessel functions product series for the computation of the modified Mathieu functions of the first kind on a high-speed digital computer, the exact cutoff wavelengths of nineteen successive modes in a hollow elliptical waveguide with an eccentricity varying from 0.00 to 0.95 are calculated. From these results it becomes clear that some important modes were forgotten in the classification until now.

A set of simple approximate formulas for the determination of the cutoff wavelength of the eight lowest order waves is proposed. The accuracy of these formulas is better than 0.25 percent. It is also proven that the bandwidth of an elliptical waveguide with a major axis equal to twice the minor axis is 25 percent less than the corresponding rectangular waveguide but nearly the double of the circular waveguide.

Some experimental results show that there exists a good agreement between the measured and calculated values of the cutoff wavelength.

Work is in progress to resolve some other important problems of wave propagation in elliptical waveguides. Among those we mention the study of the field configuration and attenuation of the lowest order modes. The design of a mode chart for elliptical resonant cavities has already been done and the results will be published soon.

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On Plane and Quasi-Optical Wave Propagation in Gyromagnetic Media

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Abstract—To promote the development and understanding of microwave magnetic devices, especially in the millimeter and submillimeter range utilizing quasi-optical techniques, a discussion of propagation and polarization of plane waves and narrow rays in gyromagnetic media in an arbitrary direction is considered. It is assumed that the medium can be described by a permeability tensor of the Polder type. The approach is structured after classical crystal optics but yields significantly different results since each of the two permitted rays is elliptically polarized. The ellipticities are derived. The phase surfaces are discussed for the lossless case. There are no optical axes but ranges of forbidden directions exist for one or both rays. *D*, *B*, and the wave vector *n* form an orthogonal set at all times. *H* is confined to the *B*, *n* plane; it gyrates along an ellipse such that the Poynting vector traces in time an elliptical cone which contains the wave vector as one mantle line. Therefore, a narrow ray can be understood to proceed along a helical path.

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I. Introduction

Many particular solutions of Maxwell's equations in magnetically biased ferrites are known. To name only a few, one may recall the work of Walker [1] on the resonances of ellipsoids or the theory of the junction circulator as derived by Bosma [2], [3] and reinterpreted by Fay and Comstock [4]. In addition, several authors have considered the propagation of waves in gyrotropic media [5]–[8], [16], [17].

The goal of this paper is to present a more fundamental treatment of wave propagation in the magnetically gyrotropic medium. The approach taken here was inspired in part by the chapter on crystal optics in Sommerfeld's *Optics* [9]. The gyrotropic medium, indeed, is closely related to a crystal, and a detailed understanding of the general properties of waves can be developed along similar lines. A self-contained approach to the topic is attempted rather than an isolated presentation of our new results.

This approach is to some extent paralleled by considerable work describing wave propagation in mag-