FIBER OPTICS

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Lecture: 5

The Wave Model of Light

The ray-model of light treats light as a beam of rays and successfully explains a few basic phenomena related to the propagation of light inside a dielectric waveguide such as an optical fiber. But these explanations are more of a qualitative nature and perhaps are not conclusive. To get a better insight into the finer aspects of propagation of light inside an optical fiber and also to understand them qualitatively as well as quantitatively, we have to refer to a more advanced model of light which is known as the “Wave-Model”.

Wave-Model of light treats light as a transverse electromagnetic wave. Then the propagation of light inside an optical fiber is explained in terms of the propagation of an electromagnetic wave inside a bound medium like the optical fiber which is a cylindrical dielectric waveguide. The purpose of using this model is to find out the relationship between the wavelength of light and its phase constant, so that we can then investigate the velocity of different modes inside the optical fiber. But prior to this analysis, let us first adopt a particularly suitable co-ordinate system to make the analysis simpler.

Since the optical fiber is a form of cylindrical dielectric waveguide, it would be very wise to choose the cylindrical co-ordinate system for our analysis. The figure 5.1 below shows the cylindrical co-ordinate system that we shall adopt for our analysis.

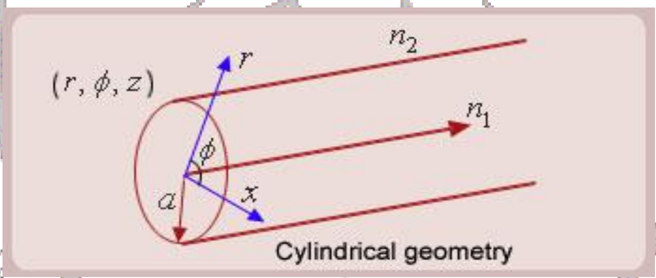

Figure 5.1: Cylindrical Co-ordinate system

From the basics of electromagnetic wave theory we already know that if n_1 and n_2 are the refractive indices of core and cladding respectively, then

$$\text{Dielectric constant of the core material}, \epsilon_1 = \epsilon_0 n_1^2$$
$$\text{Dielectric Constant of the cladding material}, \epsilon_2 = \epsilon_0 n_2^2$$

$$\text{where}, \epsilon_0 = \text{Free Space Permittivity}$$
$$\mu_1 = \mu_2 = \mu_0 (\text{Free space permeability})$$

For more simplicity of analysis, the cladding is assumed to be infinitely large in comparison to the wavelength of the light under study. The analysis then reduces to calculations across only one interface which is the core-cladding interface. The co-ordinates of any point in the above system is of the form *(r, ϕ, z)*, where ‘r’ is the radial distance of the point from the axis of the fiber, ‘ϕ’ is the angle between the meridional plane containing the point and a reference meridional plane and 'z' is the depth of the point into the fiber core as shown in the above figure.

With these assumptions, let us now pop up a problem statement for our analysis. Let us investigate the nature of the fields that exists inside an optical fiber core when light energy propagates through the fiber. For this we presently ignore the source of electromagnetic energy and also assume the core to be a perfectly source free dielectric material. Whenever we encounter such a problem statement in electromagnetics, we always solve the Maxwell's equations subject to the given constraints of the problem. Maxwell's equations for electric and magnetic fields in a source free medium can be written as:

(a) $\vec{\nabla} \cdot \vec{D} = 0$; $\vec{D}$ = Electric Displacement Vector

(b) $\vec{\nabla} \cdot \vec{B} = 0$; $\vec{B}$ = Magnetic Flux Density

(c) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$; $\vec{B} = \mu \vec{H}$; $\vec{E}$ = Electric Field

(d) $\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$; $\vec{D} = \epsilon \vec{E}$; $\vec{H}$ = Magnetic Field

From the above equations we find that the equations (c) and (d) are coupled and so, our first step would be to de-couple these two equations so as to obtain independent expressions for electric and magnetic fields and then subject them to the given limitations and conditions. The final expressions for the two fields then represent the nature of the fields in the medium under investigation.

If we substitute the relation $\vec{D} = \epsilon \vec{E}$ in equation (a), we obtain

$$\vec{\nabla} \cdot (\epsilon \vec{E}) = 0$$

Since the medium is homogeneous, ϵ is independent of space and so

$$\vec{\nabla} \cdot \vec{E} = 0$$

Similarly, since the fiber core material can be assumed to be a perfect dielectric, we obtain from equation (b):

$$\vec{\nabla} \cdot \vec{H} = 0$$

For de-coupling equations (c) and (d), we take the curl of each equation separately and then substitute one equation into the other. When we take the curl of equation (c), we get

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\vec{\nabla} \mathrm{X} \frac{\partial \vec{D}}{\partial t}$$

$$\Rightarrow \vec{\nabla} \times \vec{\nabla} \times \vec{\mathrm{E}} = -\frac{\partial}{\partial \mathrm{t}} \vec{\nabla} \times (\in \vec{\mathrm{E}})$$

$$\Rightarrow \vec{\nabla} \times \vec{\nabla} \times \vec{\mathrm{E}} = -\in \frac{\partial}{\partial \mathrm{t}} (\vec{\nabla} \times \vec{\mathrm{E}})$$

Substituting the value of $(\vec{\nabla} \times \vec{\mathrm{E}})$ from equation (d) we get

$$\Rightarrow \vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\mu\epsilon \frac{\partial^2}{\partial t^2} \vec{E} \tag{5.2}$$

From the basic vector identities we know that, for any vector $\vec{A}$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

If we use this identity in equation (5.1), we obtain

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu\epsilon \frac{\partial^2}{\partial t^2} \vec{E}$$

$$\Rightarrow 0 - \nabla^2 \vec{E} = -\mu\epsilon \frac{\partial^2}{\partial t^2} \vec{E} \qquad (\text{Since,} \vec{\nabla} \cdot \vec{E} = 0)$$

$$\Rightarrow \boxed{\boldsymbol{\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2}{\partial t^2} \vec{E}}} \tag{5.2}$$

Similarly, if we perform similar operations to equation (d) above, we would obtain a similar expression for magnetic field too. That is,

$$\boxed{\boldsymbol{\nabla^2 \vec{H} = \mu\epsilon \frac{\partial^2}{\partial t^2} \vec{H}}} \tag{5.3}$$

The two equations (5.2) and (5.3) are called the basic Wave Equations. Thus it shows that when we consider time varying electric and magnetic fields, they together constitute a wave phenomenon in the medium under study. In order to investigate the behaviour of electric and magnetic fields inside the core of an optical fiber, we have to solve the above wave equations to get the expressions for electric and magnetic fields by applying the proper boundary conditions. In other words, we have to conglomerate all the knowledge and understanding that we have, based on the ray model, and then apply it to find out the nature and characteristics of the electromagnetic fields that can exist inside the core of the optical fiber.

Since the two wave equations are similar to each other we may hence write a general expression for a wave equation involving any general vector of magnitude V in the cylindrical co-ordinate system as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \emptyset^2} + \frac{\partial^2 V}{\partial z^2} = \mu\epsilon \frac{\partial^2 V}{\partial t^2} \tag{5.4}$$

We all know that the electric field and the magnetic fields are vector quantities which, in general, have three orthogonal components. Thus, in total, there are six components to be determined for the fields to get a detailed behaviour of the fields inside the optical fiber core. This derivation, though looks cumbersome and complicated at the very first glance, is not so tedious. The reason behind this is very simple. Since there are four Maxwell's equations that are satisfied simultaneously by all the six components of the fields, so it can be clearly said that all these components are not completely independent. There must be some kind of inter-relation between them. Hence in order to simplify matters we solve the Maxwell's equations for any two components of the fields, which we assume to be independent, and then try to express the other components in terms of these components. If we choose two transverse field components there is nothing special about the transverse components because at a single point there may be an infinite number of possible components and as such there may infinite number of possible solutions. The best choice of the two components would then be, to choose the components in the direction of the net propagation of electromagnetic energy. These components are hence called as longitudinal components. Since in our analysis we have assumed the 'z' direction as the direction of propagation of net electromagnetic energy, we find out the electric field component E_z and magnetic field component H_z and then try to express the other components in terms of E_z and H_z. Once we determine the expressions for these two longitudinal components, we can then obtain the remaining four components in terms of these components by simple substitution in the following equations:

$$\boldsymbol{E_r} = \frac{-\boldsymbol{j}}{\boldsymbol{q^2}}\left\{\boldsymbol{\beta}\frac{\partial \boldsymbol{E_z}}{\partial \boldsymbol{r}} + \frac{\boldsymbol{\mu\omega}}{\boldsymbol{r}}\frac{\partial \boldsymbol{H_z}}{\partial \emptyset}\right\}$$

$$\boldsymbol{E_\emptyset} = \frac{-\boldsymbol{j}}{\boldsymbol{q^2}}\left\{\frac{\boldsymbol{\beta}}{\boldsymbol{r}}\frac{\partial \boldsymbol{E_z}}{\partial \emptyset} - \boldsymbol{\mu\omega}\frac{\partial \boldsymbol{H_z}}{\partial \boldsymbol{r}}\right\}$$

$$\boldsymbol{H_r} = \frac{-\boldsymbol{j}}{\boldsymbol{q^2}}\left\{\boldsymbol{\beta}\frac{\partial \boldsymbol{H_z}}{\partial \boldsymbol{r}} - \frac{\boldsymbol{\omega\epsilon}}{\boldsymbol{r}}\frac{\partial \boldsymbol{E_z}}{\partial \emptyset}\right\}$$

$$\boldsymbol{H_\emptyset} = \frac{-\boldsymbol{j}}{\boldsymbol{q^2}}\left\{\frac{\boldsymbol{\beta}}{\boldsymbol{r}}\frac{\partial \boldsymbol{H_z}}{\partial \emptyset} + \boldsymbol{\omega\epsilon}\frac{\partial \boldsymbol{E_z}}{\partial \boldsymbol{r}}\right\}$$

Here $\boldsymbol{q^2 = \omega^2\mu\epsilon - \beta}$ **;** ω=Angular Frequency of the launched light; β= Phase constant of the material of the core.

Few significant notions that can be observed in the expressions for the above transverse components are:

(i) Each of the transverse components are expressed in terms of derivatives of the longitudinal components E_z and H_z.

(ii) The transverse components of the electric field and the magnetic field exist even if any one of the longitudinal components vanishes.

(iii) But if both the longitudinal components vanish together all the field components vanish indicating the absence of any field in the core. In other words, if electromagnetic energy has to propagate along a fiber, the fields should have atleast one longitudinal component. That is Transverse Electromagnetic Modes (TEM) are not possible inside the core of th optical fiber.

The observation (ii) gives three distinct types of field distribution which lead us to three different modes in which electromagnetic waves can travel inside an optical fiber core. That is,

(a) If $E_z=0$, there is no longitudinal component of electric field in the direction of net propagation of electromagnetic energy. The electric field, in this case is transverse to the direction of the propagation of the wave at every point. These types of waves are, hence, called as Transverse Electric waves and the mode of propagation of these waves is said to transverse electric mode (TE).

(b) If $H_z=0$, then there is no longitudinal component of magnetic field in the direction of net propagation of electromagnetic energy. The magnetic field, in this case is transverse to the direction of the propagation of the wave at every point. These types of waves are, hence, called as Transverse Magnetic waves and the mode of propagation of these waves is said to transverse Magnetic mode (TM).

(c) If $E_z \neq 0$ and $H_z \neq 0$, the resultant mode contains all the six components of the fields and are called hybrid modes of wave propagation.

Transverse electric and transverse magnetic modes are related to propagation of meridional rays; whereas hybrid modes are related to the propagation of skew rays as we had seen earlier. This phenomenon of exhibition of three different modes of propagation was also explained by the ray model of light. With this backdrop of information, let us proceed to solve the equation to determine the expressions for the longitudinal components E_z and H_z.

For solving the wave equation for E_z and H_z let us assume a scalar quantity 'ψ' which may represent any one of the two components. This assumption of a single scalar is made in order to simplify the analysis because the wave equations for both the field components are identical and so solution for one will be identical to the solution of the other. Thus assuming a single scalar for both the fields and then replacing the scalar by the appropriate field in the final solution seems to be easier than solving separately for each component. Thus the wave equation in terms of this scalar will look like equation (5.4) if we replace V by 'ψ'.

That is:

$$\boldsymbol{\nabla^2\psi} \quad = \quad \boldsymbol{\mu\epsilon\frac{\partial^2}{\partial t^2}\psi}$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\emptyset^2}+\frac{\partial^2\psi}{\partial z^2} = \mu\epsilon\frac{\partial^2\psi}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\emptyset^2}+\frac{\partial^2\psi}{\partial z^2} = \mu\epsilon\frac{\partial^2\psi}{\partial t^2} \qquad (5.5)$$

If we assume all the field components to be time-harmonic with an angular frequency of ‘ω’ then differentiating the quantity ‘ψ’ with respect to time we would obtain the following:

$$\boldsymbol{\psi\sim e^{j\omega t}} \qquad \text{(Since } \psi \text{ is time-harmonic)}$$

Therefore

$$\frac{\partial\psi}{\partial t} \equiv j\omega\psi$$

And

$$\frac{\partial^2\psi}{\partial t^2} \equiv -\omega^2\psi$$

This shows that the second derivative of a time harmonic quantity with angular frequency ‘ω’ would be equivalent to multiplying that quantity by ‘$-\omega^2$’. Substituting the above derivative in equation (5.5) we obtain:

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\emptyset^2}+\frac{\partial^2\psi}{\partial z^2} = -\omega^2\mu\epsilon\psi \qquad (5.6)$$

Equation (5.6) is a partial differential equation that can be solved by the method of separation of variables. Since the differential equation is made up of three independent variables ‘r’, ‘ϕ’ and ‘z’, ‘ψ’ has to be a function of the three independent variables. Let us assume the solution of the above equation to be of the form:

$$\boldsymbol{\psi = R(r)\emptyset(\phi)Z(z)} \qquad (5.7)$$

Since we are interested in investigating here a travelling mode along the +z direction (along the axis of the fiber) the field should have z variation as $e^{-j\beta z}$, where β is the modal propagation constant, yet to be determined. We, therefore, have:

$$\boldsymbol{Z(z) = e^{-j\beta z}} \qquad (5.8)$$

If we differentiate equation (5.7) with respect to ‘z’ we obtain:

$$\frac{\partial\psi}{\partial z} \equiv -j\beta\psi\ ;\ \Rightarrow \frac{\partial^2\psi}{\partial z^2} \equiv -\beta^2\psi$$

Let us now consider a point (r, ϕ, z) in the optical fiber core. Keeping r and z fixed, if we vary ϕ, the point moves along a circle in a plane transverse to the axis of the fiber core (z-direction). For a change in ϕ by multiples of 2π we reach to the same point (r, ϕ, z), i.e. (r, ϕ, z) ≡ (r, ϕ+2mπ, z); m is an integer. Consequently, we should have

$$\psi(r,\phi,z) \equiv \psi(r,\phi+2m\pi,z)$$

This can be achieved if we choose

$$\emptyset(\boldsymbol{\phi}) = \boldsymbol{e^{j\nu\phi}} \quad (\nu \text{ is an integer}) \qquad (5.9)$$

(Note:$\emptyset(\phi+2\pi) = e^{j\nu(\phi+2\pi)} = e^{j\nu\phi} = \emptyset(\phi)$)

Hence,

$$\frac{\partial\psi}{\partial\phi} \equiv j\nu\psi \ ; \quad \Rightarrow \frac{\partial^2\psi}{\partial\phi^2} \equiv -\nu^2\psi$$

Substituting all these derivatives in equation (5.6) we get the following expression:

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}(-\nu^2\psi) + (-\beta^2\psi) = -\omega^2\mu\epsilon\psi$$

$$\Rightarrow \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}(-\nu^2\psi) + (-\beta^2\psi) + \omega^2\mu\epsilon\psi = 0$$

$$\Rightarrow \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \left\{(\omega^2\mu\epsilon - \beta^2) - \frac{\nu^2}{r^2}\right\}\psi = 0$$

$$\Rightarrow \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \left\{(\omega^2\mu\epsilon - \beta^2) - \frac{\nu^2}{r^2}\right\}\right]\psi = 0$$

$$\Rightarrow \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \left\{(\omega^2\mu\epsilon - \beta^2) - \frac{\nu^2}{r^2}\right\}\right]R(r) = 0$$

$$\Rightarrow \frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)}{\partial r} + \left\{(\omega^2\mu\epsilon - \beta^2) - \frac{\nu^2}{r^2}\right\}R(r) = 0 \qquad (5.10)$$

Equation (5.10) is the well-known Bessel's Equation which cannot be solved in the closed form. The series solutions of the equation are called Bessel's functions. Let us define:

$$q^2 = \omega^2\mu\epsilon - \beta^2 \qquad (5.11)$$

We have a variety of solutions to the Bessel's equation depending upon the parameters 'ν ' and 'q'. 'ν ' is an integer and a positive quantity. Depending upon the choice of q i.e., real, imaginary, complex, we get different solutions to the Bessel's equation. So to choose the proper solution we must have the physical understanding of the field distribution.

For a travelling field, β is real, and for lossless media μ and ϵ are real. Therefore q^2 is also real though it could be positive or negative. The quantity q could therefore be either purely real (if q^2 is positive) or purely imaginary (if q^2 is negative). Depending upon the sign of q^2, the Bessel's equation has different solution. Equation (5.10) being a second order differential equation, has two arbitrary constants.

For $q^2>0$, the solutions are called the Bessel functions and the Neumann functions, and are denoted by $J_\nu(qr)$ and $N_\nu(qr)$ respectively. 'ν' called the order of the function and the quantity in the brackets is called the argument of the function. The general solution to equation (5.10) can be written as a linear combination of two functions as:

$$R(r) = \alpha_1 J_\nu(qr) + \alpha_2 N_\nu(qr) \tag{5.12}$$

Here α_1 and α_2 are the arbitrary constants. Figure (5.2) below shows a representation of the Bessel and Newmann functions:

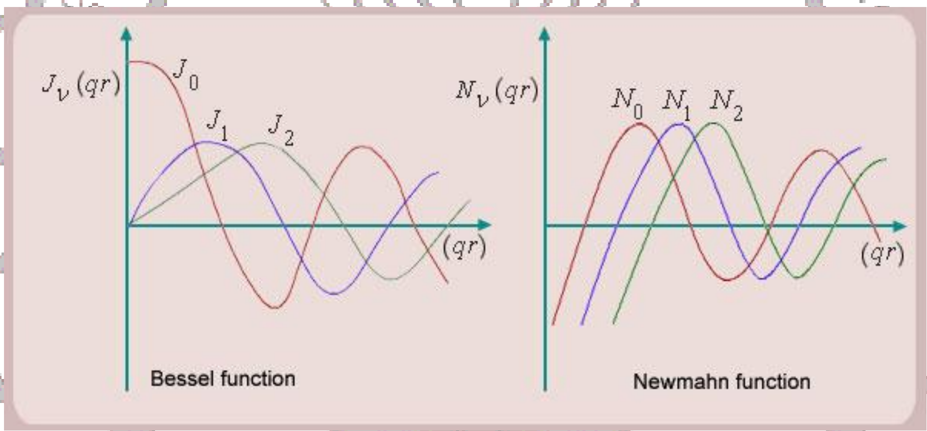

Figure 5.2: Bessel and Newmann functions.

For $q^2<0$, the solutions are called the Modified Bessel functions and are denoted by K_ν(qr/j) and I_ν(qr/j) respectively. In this case, q being purely imaginary, (qr/j) is a real quantity. The solution to equation (5.10) in this case can be written as

$$R(r) = \eta_1 K_\nu\left(\frac{qr}{j}\right) + \eta_2 I_\nu\left(\frac{qr}{j}\right) \tag{5.13}$$

Here η_1 and η_2 are the arbitrary constants.

Figure (5.3) shows the Modified Bessel functions. Since, the propagation constant β is yet undetermined, it is not clear this juncture whether q is real or imaginary and which solution to be chosen for the wave equation (5.10). In fact, this dilemma cannot be resolved unless we make use of our already gathered knowledge and understanding of the fields from the Ray-model of light.

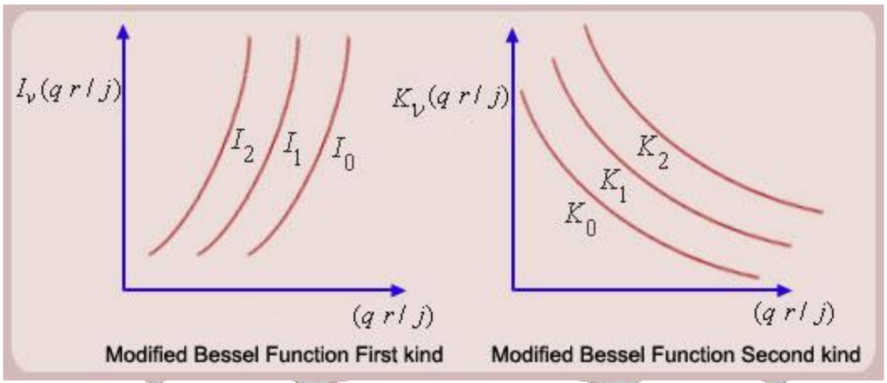

Figure 5.3: Modified Bessel Function

From figure 5.2 and 5.3 it is clear that

(1) The function $J_v(x)$ is always finite for all values of 'x'. Except $J_0(x)$ which is 1 at x=0, all other $J_v(x)$ are zero at x=0.
(2) The functions $N_v(x)$ asymptotically diverge to $-\infty$ as the argument $x \rightarrow 0$, though it is finite at all values of $x \neq 0$.
(3) Both functions $J_v(x)$ and $N_v(x)$ have oscillatory behaviour as a function of 'x'.
(4) The functions $K_v(x)$ are monotonically decreasing functions of 'x', and they go to zero as $x \rightarrow \infty$.
(5) The functions $I_v(x)$ are monotonically increasing functions of 'x', and they diverge to ∞ as $x \rightarrow \infty$.
(6) Both functions $K_v(x)$ and $I_v(x)$ are monotonic functions of 'x'.

Let us refer to the problem of guided mode propagation. We say a mode is guided when its fields are confined to the guide, and outside the guide the fields decay monotonically. Also, as seen in the wave-front investigation of the ray-model, the nature of the field pattern generated normal to the core-cladding interface was due to the superposition of the wave-fronts of the incident and the reflected rays and therefore exhibits an amplitude variation going through maxima and minima in space corresponding to constructive and destructive interference. For a guided mode we hence expect a spatially oscillatory field inside the dielectric rod, and a decaying field outside of it.

The choice of solution inside and outside the rod is quite obvious now. Inside, the rod, we should have a solution given by equation (5.12), and outside the rod solution should be given by equation (5.13). One thing to be noted here is that in saying so we have inherently put bounds on the value of β, inside and outside the core of the optical fiber.

Inside the core of the optical fiber **r < a** and $q^2>0$ and therefore $\beta^2 < \omega^2\mu\epsilon_1$

In the cladding of the optical fiber, $r > a$ and $q^2 < 0$ and therefore $\beta^2 > \omega^2\mu\epsilon_2$

Thus the propagation constant can be written in the bounded form as:

$$\omega\sqrt{\mu\epsilon_2} < \beta < \omega\sqrt{\mu\epsilon_1} \qquad (5.14)$$

Although the solution given by equation (5.12) is appropriate for (r < a) and the solution given by equation (5.13) is appropriate for (r > a), we notice here that both the terms in these equation do not correctly represent the field behaviour in the optical fiber. Since the fields are always finite at every point, so α_2 and η_2 must be zero because $N_v(x)$ and $I_v(x)$ are not finite at every point in space.

The solution to the wave equation in the core and cladding regions of the optical fiber can thus be written as

$$\boldsymbol{\psi_1(r,\phi,z,t) = \alpha_1 J_\nu(ur) e^{j\nu\phi - j\beta z + j\omega t}} \qquad \text{For } r < a \qquad (5.15)$$

$$\boldsymbol{\psi_2(r,\phi,z,t) = \eta_1 K_\nu(wr) e^{j\nu\phi - j\beta z + j\omega t}} \qquad \text{For } r > a \qquad (5.16)$$

Where we have defined

$$q \equiv u = \sqrt{\omega^2\mu\epsilon_1 - \beta^2} \qquad \text{For } r < a \qquad (5.17)$$

$$q \equiv jw = \sqrt{\beta^2 - \omega^2\mu\epsilon_2} \qquad \text{For } r > a \qquad (5.18)$$

The equations (5.15) and (5.16) give the solution of the wave equation in an optical fiber which shows the behaviour of the electromagnetic fields in the core and cladding regions of an optical fiber.

Thus the expressions for electric and magnetic fields inside the core and cladding regions can be written as:

Inside core (r < a):

$$\boldsymbol{E_{z1} = AJ_\nu(ur) e^{j\nu\phi - j\beta z + j\omega t}}$$

$$\boldsymbol{H_{z1} = BJ_\nu(ur) e^{j\nu\phi - j\beta z + j\omega t}}$$

Inside Cladding (r > a):

$$\boldsymbol{E_{z2} = CK_\nu(wr) e^{j\nu\phi - j\beta z + j\omega t}}$$

$$\boldsymbol{H_{z2} = DK_\nu(wr) e^{j\nu\phi - j\beta z + j\omega t}}$$

Here A,B,C and D are arbitrary constants and their values can be determined by substituting appropriate boundary conditions.