Gauss's law

From Wikipedia, the free encyclopedia

In physics, **Gauss's law**, also known as **Gauss's flux theorem**, is a law relating the distribution of electric charge to the resulting electric field.

The law was formulated by Carl Friedrich Gauss in 1835, but was not published until 1867.[1] It is one of the four Maxwell's equations which form the basis of classical electrodynamics, the other three being Gauss's law for magnetism, Faraday's law of induction, and Ampère's law with Maxwell's correction. Gauss's law can be used to derive Coulomb's law,[2] and vice versa.

Contents

Qualitative description of the law

In words, Gauss's law states that:

> *The net electric flux through any closed surface is equal to* $^1/_\varepsilon$ *times the net electric charge enclosed within that closed surface*.[3]

Gauss's law has a close mathematical similarity with a number of laws in other areas of physics, such as Gauss's law for magnetism and Gauss's law for gravity. In fact, any "inverse-square law" can be formulated in a way similar to Gauss's law: For example, Gauss's law itself is essentially equivalent to the inverse-square Coulomb's law, and Gauss's law for gravity is essentially equivalent to the inverse-square Newton's law of gravity.

Gauss's law is something of an electrical analogue of Ampère's law, which deals with magnetism.

The law can be expressed mathematically using vector calculus in integral form and differential form, both are equivalent since they are related by the divergence theorem, also called Gauss's theorem. Each of these forms in turn can also be expressed two ways: In terms of a relation between the electric field **E** and the total electric charge, or in terms of the electric displacement field **D** and the *free* electric charge.[4]

Equation involving E field

Gauss's law can be stated using either the electric field **E** or the electric displacement field **D**. This section shows some of the forms with **E**; the form with **D** is below, as are other forms with **E**.

Integral form

Gauss's law may be expressed as:[5]

$$\Phi_E = \frac{Q}{\varepsilon_0}$$

where Φ_E is the electric flux through a closed surface S enclosing any volume V, Q is the total charge enclosed within S, and ε_0 is the electric constant. The electric flux Φ_E is defined as a surface integral of the electric field:

$$\Phi_E = \oint\!\!\!\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A}$$

where **E** is the electric field, d**A** is a vector representing an infinitesimal element of area,[note 1] and $\cdot$ represents the dot product of two vectors.

Since the flux is defined as an *integral* of the electric field, this expression of Gauss's law is called the *integral form*.

Applying the integral form

If the electric field is known everywhere, Gauss's law makes it quite easy, in principle, to find the distribution of electric charge: The charge in any given region can be deduced by integrating the electric field to find the flux.

However, much more often, it is the reverse problem that needs to be solved: The electric charge distribution is known, and the electric field needs to be computed. This is much more difficult, since if you know the total flux through a given surface, that gives almost no information about the electric field, which (for all you know) could go in and out of the surface in arbitrarily complicated patterns.

An exception is if there is some symmetry in the situation, which mandates that the electric field passes through the surface in a uniform way. Then, if the total flux is known, the field itself can be deduced at every point. Common examples of symmetries which lend themselves to Gauss's law include cylindrical symmetry, planar symmetry, and spherical symmetry. See the article Gaussian surface for examples where these symmetries are exploited to compute electric fields.

Differential form

By the divergence theorem Gauss's law can alternatively be written in the *differential form*:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

where $\nabla \cdot \mathbf{E}$ is the divergence of the electric field, ε_0 is the electric constant, and ρ is the total electric charge density (charge per unit volume).

Equivalence of integral and differential forms

The integral and differential forms are mathematically equivalent, by the divergence theorem. Here is the argument more specifically.

Outline of proof

The integral form of Gauss's law is:

$$\oiint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{Q}{\varepsilon_0}$$

for any closed surface S containing charge Q. By the divergence theorem, this equation is equivalent to:

$$\iiint_V \nabla \cdot \mathbf{E}\ \mathrm{d}V = \frac{Q}{\varepsilon_0}$$

for any volume V containing charge Q. By the relation between charge and charge density, this equation is equivalent to:

$$\iiint_V \nabla \cdot \mathbf{E}\ \mathrm{d}V = \iiint_V \frac{\rho}{\varepsilon_0}\ \mathrm{d}V$$

for any volume V. In order for this equation to be *simultaneously true* for *every* possible volume V, it is necessary (and sufficient) for the integrands to be equal everywhere. Therefore, this equation is equivalent to:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

Thus the integral and differential forms are equivalent.

Equation involving D field

Free, bound, and total charge

The electric charge that arises in the simplest textbook situations would be classified as "free charge"—for example, the charge which is transferred in static electricity, or the charge on a capacitor plate. In contrast, "bound charge" arises only in the context of dielectric (polarizable) materials. (All materials are polarizable to some extent.) When such materials are placed in an external electric field, the electrons remain bound to their respective atoms, but shift a microscopic distance in response to the field, so that they're more on one side of the atom than the other. All these microscopic displacements add up to give a macroscopic net charge distribution, and this constitutes the "bound charge".

Although microscopically, all charge is fundamentally the same, there are often practical reasons for wanting to treat bound charge differently from free charge. The result is that the more "fundamental" Gauss's law, in terms of **E** (above), is sometimes put into the equivalent form below, which is in terms of **D** and the free charge only.

Integral form

This formulation of Gauss's law states the total charge form:

$$\Phi_D = Q_{\mathrm{free}}$$

where Φ_D is the **D**-field flux through a surface S which encloses a volume V, and Q_{free} is the free charge contained in V. The flux Φ_D is defined analogously to the flux Φ_E of the electric field **E** through S:

$$\Phi_D = \oiint_S \mathbf{D} \cdot \mathrm{d}\mathbf{A}$$

Differential form

The differential form of Gauss's law, involving free charge only, states:

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{free}}$$

where $\nabla \cdot \mathbf{D}$ is the divergence of the electric displacement field, and ρ_{free} is the free electric charge density.

Equivalence of total and free charge statements

Proof that the formulations of Gauss's law in terms of free charge are equivalent to the formulations involving total charge.

In this proof, we will show that the equation

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$$

is equivalent to the equation

$$\nabla \cdot \mathbf{D} = \rho_{\text{free}}$$

Note that we're only dealing with the differential forms, not the integral forms, but that is sufficient since the differential and integral forms are equivalent in each case, by the divergence theorem.

We introduce the polarization density **P**, which has the following relation to **E** and **D**:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

and the following relation to the bound charge:

$$\rho_{\text{bound}} = -\nabla \cdot \mathbf{P}$$

Now, consider the three equations:

$$\rho_{\text{bound}} = \nabla \cdot (-\mathbf{P})$$
$$\rho_{\text{free}} = \nabla \cdot \mathbf{D}$$
$$\rho = \nabla \cdot (\varepsilon_0 \mathbf{E})$$

The key insight is that the sum of the first two equations is the third equation. This completes the proof: The first equation is true by definition, and therefore the second equation is true if and only if the third equation is true. So the second and third equations are equivalent, which is what we wanted to prove.

Equation for linear materials

In homogeneous, isotropic, nondispersive, linear materials, there is a simple relationship between **E** and **D**:

$$\mathbf{D} = \varepsilon \mathbf{E}$$

where ε is the permittivity of the material. For the case of vacuum (aka free space), $\varepsilon = \varepsilon_0$. Under these circumstances, Gauss's law modifies to

$$\Phi_E = \frac{Q_{\text{free}}}{\varepsilon}$$

for the integral form, and

$$\nabla \cdot \mathbf{E} = \frac{\rho_{\text{free}}}{\varepsilon}$$

for the differential form.

Relation to Coulomb's law

Deriving Gauss's law from Coulomb's law

Gauss's law can be derived from Coulomb's law.

Outline of proof

Coulomb's law states that the electric field due to a stationary point charge is:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \frac{\mathbf{e_r}}{r^2}$$

where

$\mathbf{e_r}$ is the radial unit vector,

r is the radius, $|\mathbf{r}|$,

ε_0 is the electric constant,

q is the charge of the particle, which is assumed to be located at the origin.

Using the expression from Coulomb's law, we get the total field at **r** by using an integral to sum the field at **r** due to the infinitesimal charge at each other point **s** in space, to give

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{s})(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3}\, d^3\mathbf{s}$$

where ρ is the charge density. If we take the divergence of both sides of this equation with respect to **r**, and use the known theorem[7]

$$\nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|^3}\right) = 4\pi\delta(\mathbf{r})$$

where $\delta(\mathbf{r})$ is the Dirac delta function, the result is

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{\varepsilon_0} \int \rho(\mathbf{s})\ \delta(\mathbf{r}-\mathbf{s})\, d^3\mathbf{s}$$

Using the "sifting property" of the Dirac delta function, we arrive at

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon_0},$$

which is the differential form of Gauss's law, as desired.

Note that since Coulomb's law only applies to *stationary* charges, there is no reason to expect Gauss's law to hold for moving charges *based on this derivation alone*. In fact, Gauss's law *does* hold for moving charges, and in this respect Gauss's law is more general than Coulomb's law.

Deriving Coulomb's law from Gauss's law

Strictly speaking, Coulomb's law cannot be derived from Gauss's law alone, since Gauss's law does not give any information regarding the curl of **E** (see Helmholtz decomposition and Faraday's law). However, Coulomb's law *can* be proven from Gauss's law if it is assumed, in addition, that the electric field from a point charge is spherically-symmetric (this assumption, like Coulomb's law itself, is exactly true if the charge is stationary, and approximately true if the charge is in motion).

Outline of proof

Taking *S* in the integral form of Gauss's law to be a spherical surface of radius *r*, centered at the point charge *Q*, we have

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}$$

By the assumption of spherical symmetry, the integrand is a constant which can be taken out of the integral. The result is

$$4\pi r^2 \hat{\mathbf{r}} \cdot \mathbf{E}(\mathbf{r}) = \frac{Q}{\varepsilon_0}$$

where $\hat{\mathbf{r}}$ is a unit vector pointing radially away from the charge. Again by spherical symmetry, **E** points in the radial direction, and so we get

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0} \frac{\hat{\mathbf{r}}}{r^2}$$

which is essentially equivalent to Coulomb's law. Thus the inverse-square law dependence of the electric field in Coulomb's law follows from Gauss's law.

See also

- Method of image charges
- Uniqueness theorem for Poisson's equation

Notes

1. ^ More specifically, the infinitesimal area is thought of as planar and with area d*A*. The vector d**A** is normal to this area element and has magnitude d*A*.[6]

References

1. ^ Bellone, Enrico (1980). *A World on Paper: Studies on the Second Scientific Revolution*.
2. ^ Halliday, David; Resnick, Robert (1970). *Fundamentals of Physics*. John Wiley & Sons, Inc. pp. 452–53.
3. ^ Serway, Raymond A. (1996). *Physics for Scientists and Engineers with Modern Physics, 4th edition*. p. 687.
4. ^ I.S. Grant, W.R. Phillips (2008). *Electromagnetism* (2nd ed.). Manchester Physics, John Wiley & Sons. ISBN 978-0-471-92712-9.
5. ^ I.S. Grant, W.R. Phillips (2008). *Electromagnetism* (2nd ed.). Manchester Physics, John Wiley & Sons. ISBN 978-0-471-92712-9.
6. ^ Matthews, Paul (1998). *Vector Calculus*. Springer. ISBN 3-540-76180-2.
7. ^ See, for example, Griffiths, David J. (1998). *Introduction to Electrodynamics (3rd ed.)*. Prentice Hall. p. 50. ISBN 0-13-805326-X.

Jackson, John David (1998). Classical Electrodynamics, 3rd ed., New York: Wiley. ISBN 0-471-30932-X.

External links

- MIT Video Lecture Series (30 x 50 minute lectures)- Electricity and Magnetism (http://ocw.mit.edu/OcwWeb/Physics/8-02Electricity-and-MagnetismSpring2002/VideoAndCaptions/) Taught by Professor Walter Lewin.
- section on Gauss's law in an online textbook (http://www.lightandmatter.com/html_books/0sn/ch10/ch10.html#Section10.6)
- MISN-0-132 *Gauss's Law for Spherical Symmetry* (http://www.physnet.org/modules/pdf_modules/m132.pdf) (PDF file) by Peter Signell for Project PHYSNET (http://www.physnet.org).
- MISN-0-133 *Gauss's Law Applied to Cylindrical and Planar Charge Distributions* (http://www.physnet.org/modules/pdf_modules/m133.pdf) (PDF file) by Peter Signell for Project PHYSNET (http://www.physnet.org).