A New Criterion for Linear 2-Port Stability Using a Single Geometrically Derived Parameter

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***Abstract*—A new stability parameter "$\mu(S)$," is defined for linear 2-port circuits using a geometrical approach. It is shown that $\mu > 1$ alone is necessary and sufficient for a circuit to be unconditionally stable, where**

$$\mu \equiv \frac{1 - |S_{11}|^2}{|S_{22} - S_{11}^* \Delta| + |S_{21} S_{12}|}.$$

This single parameter can replace the dual Rollet ($K > 1$) and auxiliary conditions for determining unconditional stability. The parameters K and μ are compared by discussing their implications in terms of mapping circles.

I. Introduction

A LINEAR 2-port circuit is said to be absolutely, or unconditionally, stable if there is no passive source ($|\Gamma_S| < 1$) and passive load ($|\Gamma_L| < 1$) combination that can cause the circuit to oscillate. It is known [1]–[6] that the combination of the Rollet [7] condition

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12} S_{21}|} > 1 \tag{1}$$

together with *any one* of the following auxiliary conditions is necessary and sufficient for unconditional stability.

$$B_1 = 1 + |S_{11}|^2 - |S_{22}|^2 - |\Delta|^2 > 0 \tag{2a}$$

$$B_2 = 1 - |S_{11}|^2 + |S_{22}|^2 - |\Delta|^2 > 0 \tag{2b}$$

$$|\Delta| = |S_{11} S_{22} - S_{12} S_{21}| < 1 \tag{2c}$$

$$1 - |S_{11}|^2 > |S_{12} S_{21}| \tag{2d}$$

$$1 - |S_{22}|^2 > |S_{12} S_{21}|. \tag{2e}$$

Various authors have claimed that conditions (2d) and (2e) are both required [8]–[11] while others have stated that only one of the conditions is required [3]. Appendix I shows explicitly that when $K > 1$, then auxiliary condition (2d) implies (2e) and vice versa, i.e., only one auxiliary condition is needed.

The design of active circuits requires that multiple parameters be evaluated over a wide frequency range much larger than their intended pass-band. If a circuit or a device fails to meet these conditions, it is difficult to assess the degree of potential instability that exists, since the values associated with (1) and (2) provide little direct physical insight into the degree of stability or lack thereof.

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Mappings

The input and output reflection coefficient are related to the load and source reflection coefficient by the well known linear fractional transformation which maps circles into circles (where a straight line is the special case of a circle containing the point ∞) [12]

$$\Gamma_{in} = f(\Gamma_L) = S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - S_{22} \Gamma_L}, \tag{3a}$$

and

$$\Gamma_{out} = g(\Gamma_S) = S_{22} + \frac{S_{12} S_{21} \Gamma_S}{1 - S_{11} \Gamma_S}. \tag{3b}$$

The inverses, $\Gamma_L = f^{-1}(\Gamma_{in})$, and $\Gamma_S = g^{-1}(\Gamma_{out})$, are well defined provided that $S_{12} S_{21} \neq 0$

$$\Gamma_L = f^{-1}(\Gamma_{in}) = \frac{S_{11} - \Gamma_{in}}{\Delta - S_{22} \Gamma_{in}} \tag{3c}$$

$$\Gamma_S = g^{-1}(\Gamma_{out}) = \frac{S_{22} - \Gamma_{out}}{\Delta - S_{11} \Gamma_{out}}. \tag{3d}$$

Consequently, the approach in this paper is to initially assume that the circuit is not unilateral ($S_{12} S_{21} \neq 0$) and then to examine the unilateral case afterwards.

The function f and its inverse f^{-1} are mappings between complex points in the Γ_{in} plane and the Γ_L plane. Fig. 1 illustrates the domains and ranges of these maps that are necessary and sufficient for unconditional stability. A circuit is unconditionally stable if the function "f" maps the unit disk in the Γ_L-plane into the unit disk in the Γ_{in}-plane. This is equivalent to saying that the inverse f^{-1} maps the unit disk in the Γ_{in}-plane onto a region which contains the unit disk in the Γ_L-plane. Note that the unit disk is a set of complex reflection coefficient whose magnitude is less than or equal to one. This is exactly the region represented by the conventional or passive Smith Chart denoted in this paper as USC standing for *Unit Smith Chart*. Because of the circle preserving property of linear fractional transformations, the inverse mapping could typically look either like Fig. 1(b) or (c). These functional

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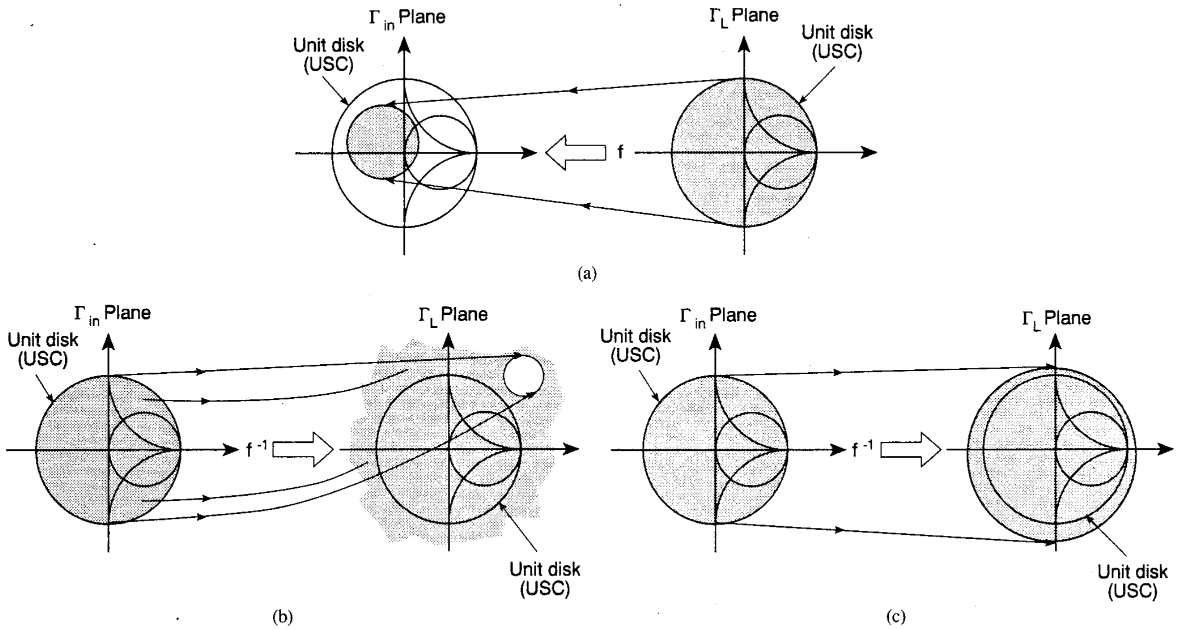


Fig. 1. Unconditional stability in terms of mapped regions.

characteristics and their analytical representation form the basis for defining the new measure of stability.

Traditional stability circles are defined [9] in terms of the mappings f and g as follows:

$$\text{Source Stability Circle} = g^{-1}(|\Gamma_{\text{out}}| = 1)$$

$$\text{Load Stability Circle} = f^{-1}(\Gamma_{\text{in}}| = 1)$$

The radius and center of these circles results in the commonly known expressions:

$$r_S = \frac{|S_{21}S_{12}|}{||S_{11}|^2 - |\Delta|^2|}$$

$$C_S = \frac{S_{11}^* - S_{22}\Delta^*}{|S_{11}|^2 - |\Delta|^2}$$

$$c_S = |C_S|$$

$$r_L = \frac{|S_{21}S_{12}|}{||S_{22}|^2 - |\Delta|^2|}$$

$$C_L = \frac{S_{22}^* - S_{11}\Delta^*}{|S_{22}|^2 - |\Delta|^2}$$

$$c_L = |C_L| \tag{4}$$

Another set of circles, referred to as input and output mapping circles can be defined in the Γ_{in} plane and Γ_{out} plane as follows:

$$\text{Input Unit Mapping Circle} = f(|\Gamma_L| = 1)$$

$$\text{Output Unit Mapping Circle} = g(|\Gamma_S| = 1)$$

The radius and center of these circles are

$$r_{\text{in}} = \frac{|S_{21}S_{12}|}{|1 - |S_{22}|^2|}$$

$$C_{\text{in}} = \frac{S_{11} - S_{22}^*\Delta}{1 - |S_{22}|^2}$$

$$c_{\text{in}} = |C_{\text{in}}|$$

$$r_{\text{out}} = \frac{|S_{21}S_{12}|}{|1 - |S_{11}|^2|}$$

$$C_{\text{out}} = \frac{S_{22} - S_{11}^*\Delta}{1 - |S_{11}|^2}$$

$$c_{\text{out}} = |C_{\text{out}}|. \tag{5}$$

The mappings illustrated in Fig. 1(b) and (c) appear to be different. However, a stereographic representation of the complex plane onto a unit sphere [13] reveals that they are inherently the same, and hence a single parameter should exist (see Appendix II). A new parameter, "μ," will be defined based upon the mapping "f." It will be shown that the value of μ alone, unambiguously determines if the circuit is unconditionally stable or potentially (conditionally) unstable. A dual parameter designated μ' can be defined based upon the mapping "g," and also uniquely determines whether the circuit is unconditionally stable. This approach also provides direct physical insight into the degree to which the Unit Smith Chart (USC), is encroached by possible unstable load and source regions, providing the engineer with a measure of the risk or margin associated with his design.

Complex Representation of a Disk (or Disk Complement)

The following inequality

$$|z|^2 - za - z^*a^* < b \tag{6}$$

where a is a complex number and b is a real number such that $|a|^2 + b \geq 0$, describes a circular disk of complex points whose center is $C = a^*$, and whose radius is

$$r = \sqrt{b + |a|^2}.$$

This is seen by adding the term $|a|^2$ to both sides of (6) and manipulating the results to get

$$|z - a^*| < \sqrt{b + |a|^2}.$$

If the "less than" sign in (6) were reversed to be a "greater than" sign then

$$|z - a^*| > \sqrt{b + |a|^2}$$

which describes a region external to the above defined disk. This external region is referred to, therefore, as a "disk complement."

II. Defining the New Stability Factor μ

The new parameter, μ, is defined as the minimum distance in the Γ_L-plane between the origin of the Unit Smith Chart and the unstable region. A negative value for this distance parameter indicates that the unstable region overlaps the origin of USC. It turns out that μ is described by a relatively simple expression whose analytical form is the same regardless of whether the inverse mapping, f^{-1}, is of the type illustrated by Fig. 1(b), or (c).

It will now be shown that the mapping "f^{-1}," illustrated in Fig. 1(b) and (c), will occur if and only if the distance $\mu(S) > 1$. This will be argued by showing that these mappings imply that $\mu(S) > 1$ and then justifying the reversibility of the steps. The above statement is equivalent to the following mathematical statement

$$[\text{USC} \subset \{\Gamma_L: \Gamma_L = f^{-1}(|\Gamma_{\text{in}}| < 1)\}] \Leftrightarrow \mu(S) > 1. \tag{7}$$

The range of the map $f^{-1}(|\Gamma_{\text{in}}| < 1)$, is determined by $|f(\Gamma_L)| < 1$. Straight forward substitution from (3a) yields

$$|\Gamma_L|^2[|S_{22}|^2 - |\Delta|^2] + \Gamma_L^*[S_{11}\Delta^* - S_{22}^*] + \Gamma_L[S_{11}^*\Delta - S_{22}] > |S_{11}|^2 - 1.$$

Dividing this expression by $|S_{22}|^2 - |\Delta|^2$, one obtains the complex variable representation of a disk or disk complement (see (6)) depending on whether or not $|S_{22}|^2 - |\Delta|^2 > 0$ or $|S_{22}|^2 - |\Delta|^2 < 0$. The resulting inequalities are as follows:

$$\left|\Gamma_L - \frac{S_{22}^* - S_{11}\Delta^*}{|S_{22}|^2 - |\Delta|^2}\right| > \frac{|S_{21}S_{12}|}{|S_{22}|^2 - |\Delta|^2} \tag{8}$$

where $|S_{22}|^2 - |\Delta|^2 > 0$, and

$$\left|\Gamma_L - \frac{S_{22}^* - S_{11}\Delta^*}{|S_{22}|^2 - |\Delta|^2}\right| < \frac{|S_{21}S_{12}|}{|\Delta|^2 - |S_{22}|} \tag{9}$$

where $|S_{22}|^2 - |\Delta|^2 < 0$.

The circle defined by replacing the inequalities in (8) and (9) with an **equal sign** is commonly referred to as a stability circle. This analysis can be carried further since (8) and (9) preserve the information about which region is the stable one.

One can consider what is required for the USC to be contained in the range of the mapping f^{-1} as illustrated in Fig. 1(b) and (c).

Case 1: $|S_{22}|^2 - |\Delta|^2 > 0$

In this case the range of our mapping in the Γ_L plane is the region outside the circle defined by (8) and must be of the type illustrated in Fig. 1(b). It is clear that the USC is contained in the disk complement if and only if

$$c - r > 1 \tag{10}$$

where

c = the distance from the center of the Smith Chart to the center of the disk complement
r = the radius of the disk complement

Substituting the values for c and r from (8) into (10),

$$\left|\frac{S_{22}^* - S_{11}\Delta^*}{|S_{22}|^2 - |\Delta|^2}\right| - \frac{|S_{21}S_{12}|}{|S_{22}|^2 - |\Delta|^2} > 1.$$

Since the denominator of the expressions, $|S_{22}|^2 - |\Delta|^2$ is positive, one can simplify to

$$\frac{|S_{22} - S_{11}^*\Delta| - |S_{21}S_{12}|}{|S_{22}|^2 - |\Delta|^2} > 1 \tag{11}$$

Case 2: $|S_{22}|^2 - |\Delta|^2 < 0$

In this case the range of the mapping "f^{-1}" is the region inside the disk defined by (9) and must be of the type illustrated in Fig. 1(c). It is clear that USC is contained in this disk if and only if

$$r - c > 1 \tag{12}$$

Substituting the values for c and r from (9) into (12),

$$\frac{|S_{21}S_{12}|}{|\Delta|^2 - |S_{22}|^2} - \left|\frac{S_{22}^* - S_{11}\Delta^*}{|S_{22}|^2 - |\Delta|^2}\right| > 1.$$

In this case the denominator, $|S_{22}|^2 - |\Delta|^2$, is negative, and removal of the absolute value sign requires a rearrangement, yielding

$$\frac{|S_{21}S_{12}|}{|\Delta|^2 - |S_{22}|^2} - \frac{|S_{22}^* - S_{11}\Delta^*|}{|\Delta|^2 - |S_{22}|^2} > 1.$$

By reordering the terms in the denominator, one arrives at the following result:

$$\frac{|S_{22} - S_{11}^*\Delta| - |S_{21}S_{12}|}{|S_{22}|^2 - |\Delta|^2} > 1. \quad (13)$$

It is important to note that (13) is *identical* to (11), and thus a *single* stability parameter emerges regardless of the value of $|S_{22}|^2 - |\Delta|^2$.

The apparent singularity presented by the denominator of (13) can be eliminated and the expression further simplified by noting that

$$|S_{22}|^2 - |\Delta|^2 = \frac{|S_{22} - S_{11}^*\Delta|^2 - |S_{21}S_{12}|^2}{1 - |S_{11}|^2}. \quad (14)$$

Factoring the numerator of (14) which is the difference of two squares, and substituting $|S_{22}|^2 - |\Delta|^2$ from (14) into (13) yields,

$$\mu \equiv \frac{1 - |S_{11}|^2}{|S_{22} - S_{11}^*\Delta| + |S_{21}S_{12}|} > 1. \quad (15)$$

It is interesting to note that the case where $|S_{22}|^2 - |\Delta|^2 = 0$ results in a stability circle which is a straight line but presents no difficulty with (15).

All steps taken above have been completely reversible, so it has been shown that the mapping "f^{-1}," illustrated in Fig. 1(b) and (c), will occur if and only if $\mu(S) > 1$.

Also note from (5) that the magnitude, $|\mu|$, is related to the output mapping circles by

$$\frac{1}{|\mu|} = r_{out} + c_{out}. \quad (16)$$

III. Proof that $\mu > 1 \Leftrightarrow$ Unconditional Stability

In order to prove that $\mu > 1$ if and only if unconditional stability exists, it must first be shown that the mapping "f" illustrated in Fig. 1(a) implies that

$$K > 1 \quad (17a)$$

and

$$1 - |S_{22}|^2 > |S_{21}S_{12}|. \quad (17b)$$

The range of the mapping $f(|\Gamma_L| < 1)$, is determined by $|f^{-1}(\Gamma_{in})| < 1$. Solving for Γ_{in}, one obtains the following,

$$|\Gamma_{in}|^2[|S_{22}|^2 - 1] + \Gamma_{in}[S_{11}^* - S_{22}\Delta^*] + \Gamma_{in}^*[S_{11} - S_{22}^*\Delta] > |S_{11}|^2 - |\Delta|^2. \quad (18)$$

It is now desirable to divide both sides of (18) by $|S_{22}|^2 - 1$. If $|S_{22}|^2 - 1 > 0$, the range of the mapping "f" would be a **disk complement** and contradictory to the assumption that the mapping is that illustrated in Fig. 1(a). Therefore, $|S_{22}|^2 - 1 < 0$. Dividing and using (6), one obtains

$$\left|\Gamma_{in} - \frac{S_{11} - S_{22}^*\Delta}{1 - |S_{22}|^2}\right| < \frac{|S_{21}S_{12}|}{[1 - |S_{22}|^2]}. \quad (19)$$

By noting that this mapping results in a disk that lies inside the USC in the Γ_{in} plane (see Fig. 1(a)), we see that $c + r < 1$. Moving r to the right hand side yields

$$\frac{|S_{11} - S_{22}^*\Delta|}{1 - |S_{22}|^2} < 1 - \frac{|S_{21}S_{12}|}{1 - |S_{22}|^2}. \quad (20)$$

Since the left side of (20) is greater than or equal to zero, then

$$|S_{21}S_{12}| < 1 - |S_{22}|^2. \quad (21)$$

Furthermore, squaring both sides of (20) and substituting

$$|S_{11} - S_{22}^*\Delta|^2 = |S_{21}S_{12}|^2 + [1 - |S_{22}|^2][|S_{11}|^2 - |\Delta|^2] \quad (22)$$

into the result yields

$$K \equiv \frac{1 - |S_{22}|^2 - |S_{11}|^2 + |\Delta|^2}{2|S_{21}S_{12}|} > 1. \quad (23)$$

All of the steps taken from (18) to (23) are completely reversible, so it has been shown that the mapping "f" illustrated in Fig. 1(a) implies that

$$K > 1$$
$$1 - |S_{22}|^2 > |S_{21}S_{12}|.$$

This is exactly the two conditions of (1) and (2e) which are known to be necessary and sufficient for unconditional stability of a linear 2-port. Thus it has been shown that $\mu > 1$ if and only if a 2-port network is unconditionally stable.

We now look at the unilateral case of μ. It is clear by substitution that

$$\mu(\text{unilateral}) = \frac{1 - |S_{11}|^2}{|S_{22}|\,|1 - |S_{11}|^2|} \quad (24)$$

so it is immediately obvious that $\mu > 1$ if and only if $|S_{22}| < 1$ and $|S_{11}| < 1$, which are the necessary and sufficient conditions for unconditional stability of a unilateral circuit.

IV. Definition of the Dual Parameter μ'

Another parameter, μ', can be defined based on the mapping function "g" in (3) and likewise $\mu' > 1$ if and only if a 2-port network is unconditionally stable. The dual parameter is given by

$$\mu' \equiv \frac{1 - |S_{22}|^2}{|S_{11} - S_{22}^*\Delta| + |S_{21}S_{12}|} > 1. \quad (25)$$

This further implies that $\mu(S) > 1 \Leftrightarrow \mu'(S) > 1$. Also,

$$\frac{1}{|\mu'|} = r_{in} + c_{in} \quad (26)$$

V. Geometric Implications of $K > 1$ Versus $\mu > 1$

The criterion $\mu > 1$ has been shown to be necessary and sufficient for unconditional stability. It is well known that $K > 1$ is only a necessary condition. A better understanding of these criteria can be seen by examining their geometric implications in terms of mapping circles.

The following proof will show that $K > 1$ is geometrically equivalent to stating that the input and output mapping circles and the source and load stability circles do not intersect the boundary of the USC. Although Meys [5] has proven this using a Y-parameter formulation, the following S-parameter approach is useful in seeing the geometrical relationship between K and μ. $K > 1$ implies

$$1 - |S_{22}|^2 > |S_{11}|^2 - |\Delta|^2 + 2|S_{21}S_{12}|. \qquad (27)$$

Since the left hand side of the inequality is the denominator term in the input mapping circle expressions (5), it is desirable to divide the inequality (27) by $1 - |S_{22}|^2$. However, division requires a knowledge of the sign of this term. Since this unknown, one must consider both cases, i.e., when the quantity is positive and when it is negative.

Case A: $1 - |S_{22}|^2 > 0$

In this case division results in

$$1 > \frac{|S_{11}|^2 - |\Delta|^2 + 2|S_{21}S_{12}|}{1 - |S_{22}|^2}.$$

Multiplying the numerator and denominator of the right hand side by $1 - |S_{22}|^2$ and substituting the identity

$$(1 - |S_{22}|^2)(|S_{11}|^2 - |\Delta|^2) = |S_{11} - S_{22}^*\Delta|^2 - |S_{21}S_{12}|^2$$

results in

$$\left(1 - \frac{|S_{21}S_{12}|}{1 - |S_{22}|^2}\right)^2 > \frac{|S_{11} - S_{22}^*\Delta|^2}{(1 - |S_{22}|^2)^2}. \qquad (28)$$

Applying a square root operation to the inequality (28) requires that consideration be given to the fact that the left hand side could be positive or negative. Each of these possibilities is handled separately.

Case A_1: $1 - |S_{22}|^2 > |S_{21}S_{12}|$

In this case (28) becomes

$$\frac{|S_{11} - S_{22}^*\Delta|}{1 - |S_{22}|^2} + \frac{|S_{21}S_{12}|}{1 - |S_{22}|^2} < 1.$$

From (5) this means that the radius and center of the input mapping circles satisfy

$$c_{in} + r_{in} < 1$$

This is the case of unconditional stability and is illustrated in Fig. 2(a).

Case A_2: $1 - |S_{22}|^2 < |S_{21}S_{12}|$

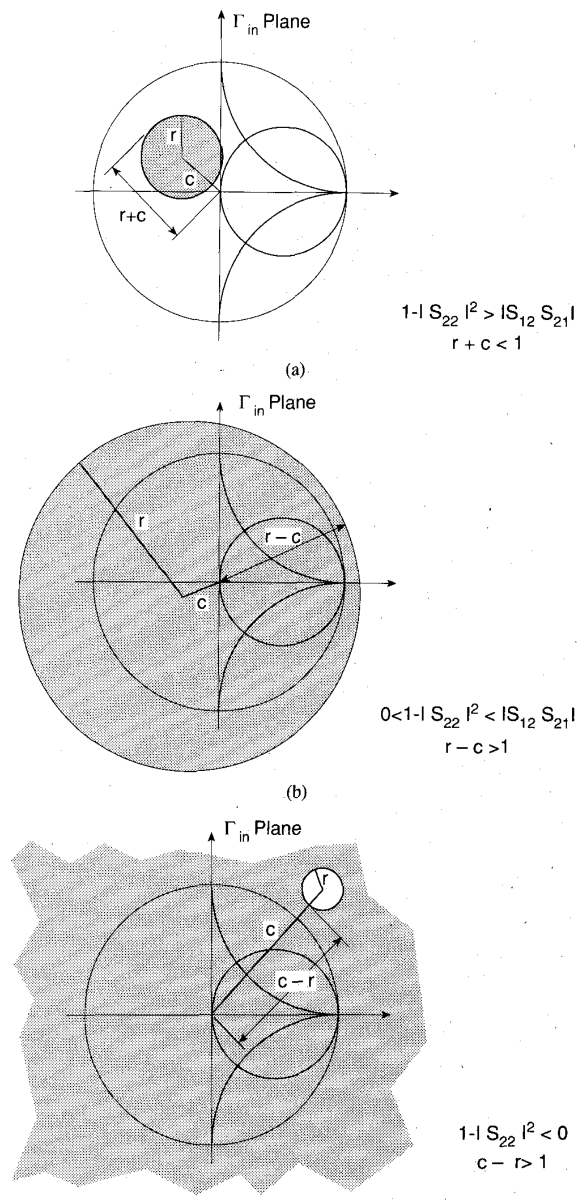


Fig. 2. Illustration of possible mapping circles for $K > 1$.

In this case, simple algebraic manipulation results in

$$\frac{|S_{21}S_{12}|}{1 - |S_{22}|^2} - \frac{|S_{11} - S_{22}^*\Delta|}{1 - |S_{22}|^2} > 1.$$

From (5),

$$r_{in} - c_{in} > 1$$

resulting in the case illustrated by Fig. 2(b). Clearly, the input mapping circle does not intersect the boundary of the USC, and the circuit is potentially unstable since the mapped region contains values of Γ_{in} whose magnitude exceeds unity.

Case B: $1 - |S_{22}|^2 < 0$.

In this case division of (27) and manipulations similar to those of Case A results in

$$\left(1 + \frac{|S_{21}S_{12}|}{|S_{22}|^2 - 1}\right)^2 < \frac{|S_{11} - S_{22}^*\Delta|^2}{(|S_{22}|^2 - 1)^2}.$$

The square root operation can be unambiguously performed in this case since the quantities inside the brackets are known to be positive. After performing the square root operation, a rearrangement of terms yields

$$\frac{|S_{11} - S_{22}^*\Delta|}{|S_{22}|^2 - 1} - \frac{|S_{21}S_{12}|}{|S_{22}|^2 - 1} > 1$$

$$c_{in} - r_{in} > 1$$

Again the input mapping circles does not intersect the boundary of the USC (see Fig. 2(c)), and also, the circuit is potentially unstable.

Interchanging the subscripts, ($2 \rightarrow 1$, and $1 \rightarrow 2$) in the previous argument and replacing r_{in} and c_{in} with r_{out} and c_{out}, reveals that the output mapping circle also does not intersect the USC, and the same three *identical* scenarios illustrated in Fig. 2 will occur in the Γ_{out} plane as well. Since the function "f" and "g" are one to one mappings, the *output and input mapping circles intersect* the boundary of the USC if and only if the *source and load stability circles intersect* the boundary of the USC. Therefore, $K > 1$ is equivalent to the nonintersection of stability circles with the boundary of the USC.

The condition $K > 1$ implies that one of three possible mapping of the USC into the Γ_{in} or Γ_{out} plane can occur. The parameter K can be compared to the new stability parameter μ by discussing the implications of $\mu > 1$ geometrically in terms of mapping circles. From (16), if $|\mu| > 1$, then $r_{out} + c_{out} < 1$, and the output mapping circle lies inside the USC. This circle is the image of the USC boundary from the Γ_S plane. However, $|\mu|$ does not provide enough information to determine whether the USC in the Γ_S plane is transformed to the interior of the mapping circle (a disk) or the external region of the mapping circle (a disk complement). The sign of μ resolves this ambiguity. If μ is positive, the mapped region is disk and the circuit is unconditionally stable, otherwise the mapped region is a disk complement permitting reflection coefficients of unlimited magnitude (i.e., conditionally unstable).

This can be contrasted to the ambiguity that results when only the condition $K > 1$ is known. With only this information three possible mapping scenarios are implied and only one is correct (Fig. 2). One of the scenarios corresponds to unconditional stability, while the other two correspond to potential instability. An auxiliary condition (2) is, therefore, required to determine which of these three scenarios is the correct one. Accordingly, it can be seen geometrically why $\mu > 1$ is equivalent to unconditional stability. Although $K > 1$, alone, is not equivalent to unconditional stability, it does imply the nonintersection of the USC boundary with mapping and stability circles. Furthermore, it specifies possible mapped regions associated with each of the three scenarios.

VI. Examples

A table has been constructed to compare the value of K, $|\Delta|$, B_1, and B_2 to the two new stability criteria, μ, and μ'. The following nine sets of 2-port S-parameters have been used to calculate the stability factors:

1. *Unconditional Stability*

$$S_{11} = .20 \angle 20^\circ \quad S_{21} = 3 \angle 40^\circ$$
$$S_{12} = .05 \angle 120^\circ \quad S_{22} = .5 \angle -50^\circ$$

2. *Conditionally Unstable: $K > 1$ and $B_1 < 0$*

$$S_{11} = .75 \angle -60^\circ \quad S_{21} = 6 \angle 90^\circ$$
$$S_{12} = .3 \angle 70^\circ \quad S_{22} = .5 \angle 60^\circ$$

3. *Conditionally Unstable: $K < 1$ and $B_1 > 0$*

$$S_{11} = 1.05 \angle 20^\circ \quad S_{21} = 3 \angle 40^\circ$$
$$S_{12} = .05 \angle 120^\circ \quad S_{22} = .5 \angle -50^\circ$$

4. *Unconditionally Stable: Unilateral Case*

$$S_{11} = .10 \angle 0^\circ \quad S_{21} = 0 \angle 0^\circ$$
$$S_{12} = 0 \angle 0^\circ \quad S_{22} = 0.3 \angle 0^\circ$$

5. *Input Unstable: Unilateral Case*

$$S_{11} = 1.2 \angle 0^\circ \quad S_{21} = 0 \angle 0^\circ$$
$$S_{12} = 0 \angle 0^\circ \quad S_{22} = 0.3 \angle 0^\circ$$

6. *Output Unstable: Unilateral Case*

$$S_{11} = .10 \angle 0^\circ \quad S_{21} = 0 \angle 0^\circ$$
$$S_{12} = 0 \angle 0^\circ \quad S_{22} = 1.3 \angle 0^\circ$$

7. *Unconditionally Stable: $|\Delta|^2 = |S_{22}|^2$ Straight Line Stability Curve*

$$S_{11} = .5 \angle 0^\circ \quad S_{21} = 2 \angle 0^\circ$$
$$S_{12} = 0.25 \angle 180^\circ \quad S_{22} = 0.1 \angle 0^\circ$$

8. *Conditionally Unstable:* NEC710 at 2 GHz

$$S_{11} = 0.95 \angle -22^\circ \quad S_{21} = 3.5 \angle 165^\circ$$
$$S_{12} = 0.04 \angle 80^\circ \quad S_{22} = 0.61 \angle -13^\circ$$

9. *Unconditionally Stable:* NEC710 at 18 GHz

$$S_{11} = 0.69 \angle -123^\circ \quad S_{21} = 1.29 \angle 78^\circ$$
$$S_{12} = 0.11 \angle 48^\circ \quad S_{22} = 0.52 \angle -77^\circ$$

The stability factors for the nine sets of S-Parameters listed above have been tabulated in the table below:

TABLE I
STABILITY PARAMETER COMPARISON

S-Parameter Set	K	$\|\Delta\|$	B_1	B_2	μ	μ'
1	2.5735	0.2491	0.7280	1.1480	1.5987	3.3004
2	1.3435	2.1562	−3.3367	−3.9617	0.1485	0.3381
3	0.3358	0.6732	1.3993	−0.3057	−0.2862	0.8683
4	∞	.03	.9191	1.0791	3.333	10
5	−∞	0.36	2.2204	−0.4796	−3.333	0.8333
6	−∞	0.130	−0.6969	2.6631	0.7692	−10.0
7	7.5	0.10	1.23	0.75	7.5	1.833
8	0.1880	0.5721	1.2032	0.1424	0.3307	0.8294
9	1.1203	0.2539	1.1412	0.7298	1.0484	1.0305

VI. CONCLUSIONS

It has been shown that a single parameter, μ, exists that is necessary and sufficient to show unconditional stability of any 2-port network. A companion parameter, μ', also exists and is necessary and sufficient to show unconditional stability of any 2-port as well. Although the condition $K > 1$ implies that the mapping and stability circles do not intersect the boundary of the USC, an ambiguity involving three possible mapped regions, not all unconditionally stable, results, and thus requires an auxiliary condition for resolution. No such ambiguity occurs with the μ (or μ') approach. A comparison of the new stability parameter (μ or μ') for S-parameter values that satisfy or violate the traditional stability conditions (1) and (2) has been illustrated.

APPENDIX I

It will be shown that if $K > 1$ and $1 - |S_{22}|^2 > |S_{21}S_{12}|$, then $1 - |S_{11}|^2 > |S_{21}S_{12}|$. Expanding Δ and rearranging terms in (20) yields

$$|S_{11}|^2(1 - |S_{22}|^2) + S_{11}(S_{12}^*S_{21}^*S_{22}) + S_{11}^*(S_{12}^*S_{21}^*S_{22})^* < 1 - |S_{22}|^2 - 2|S_{12}S_{21}| + |S_{12}S_{21}|^2.$$

Dividing by the positive quantity $1 - |S_{22}|^2$ yields

$$|S_{11}|^2 + S_{11}\frac{(S_{12}^*S_{21}^*S_{22})}{1 - |S_{22}|^2} + S_{11}^*\frac{(S_{12}^*S_{21}^*S_{22})^*}{1 - |S_{22}|^2} < \frac{|S_{12}S_{21}|^2 - 2|S_{21}S_{12}| + 1 - |S_{22}|^2}{1 - |S_{22}|^2}.$$

From (4), one notes that this is the equation for a disk in the S_{11} plane and can be expressed as follows

$$\left|S_{11} + \frac{S_{12}S_{21}S_{22}^*}{1 - |S_{22}|^2}\right| < \frac{1 - |S_{22}|^2 - |S_{21}S_{12}|}{1 - |S_{22}|^2}.$$

It is clear that S_{11} can only take on values that fall inside this disk. Such value must obey the following inequality

$$|S_{11}| < r + c$$

where r is the radius of the disk in the S_{11} plane and c is the *magnitude* of the center of the disk in the S_{11} plane. This results in

$$|S_{11}| < \frac{-|S_{12}S_{21}|}{1 + |S_{22}|} + 1. \tag{29}$$

Since $1 - |S_{22}|^2 > |S_{21}S_{12}|$,

$$|S_{22}| < \sqrt{1 - |S_{12}S_{21}|}. \tag{30}$$

Substituting for $|S_{22}|$ in (29) with (30),

$$|S_{11}| < \frac{-|S_{12}S_{21}|}{1 + \sqrt{1 - |S_{12}S_{21}|}} + 1.$$

Simplifying, one obtains

$$|S_{11}| < \sqrt{1 - |S_{12}S_{21}|}$$

and therefore

$$1 - |S_{11}|^2 > |S_{21}S_{12}|$$

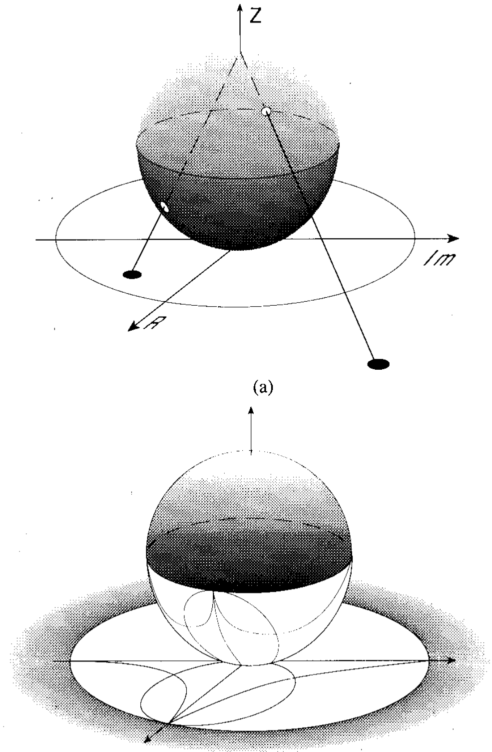
as was to be shown.

In order to show the converse one can repeat this proof by substituting S_{22} for S_{11} and visa versa. The result is the proof that if $K > 1$ and $1 - |S_{11}|^2 > |S_{21}S_{12}|$, then $1 - |S_{22}|^2 > |S_{21}S_{12}|$. If $K > 1$, then only one of the auxiliary conditions (2) is needed for the unconditional stability.

APPENDIX II

For an unconditionally stable circuit it would appear that two different stability circle configurations are possible as illustrated in Fig. 1(b) and (c). That these two distinct cases are really the same, provided the motivation for recognizing that a single stability parameter μ was possible, as can be seen by examining the stereographic representation of the complex plane as a sphere.

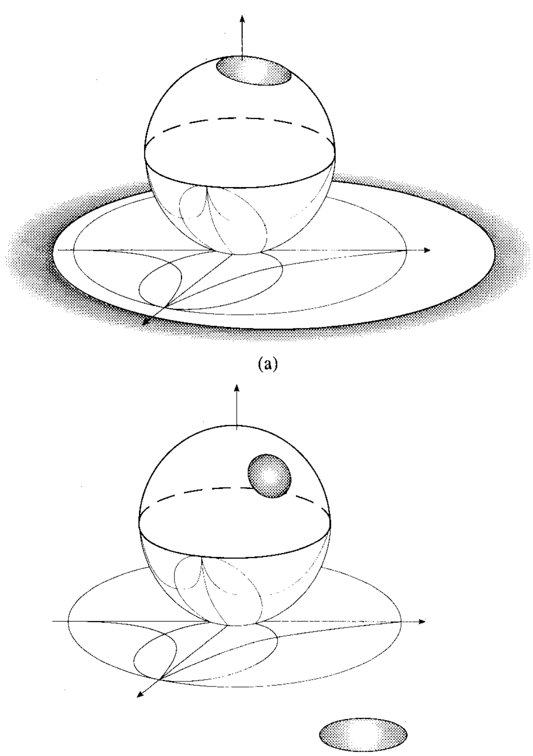
Stereographic projection [13] is accomplished by placing a sphere with unity diameter on the complex plane. As illustrated in Fig. 3(a), the south pole is located at the origin and the north pole on the z axis, perpendicular to the plane. Points on the sphere are identified with points on the plane by projecting a line from the north pole through the sphere to the plane. Therefore, a circle of unit radius in the plane is equivalent to the equator of the sphere. In general circles in the plane are transformed into circles on the sphere, and straight lines in the plane be-



(a)

(b)

Fig. 3. Stereographic projection.



(a)

(b)

Fig. 4. Stereographic projection of stability circles.

come circles on the sphere that pass through the north pole. Therefore, from the point of view of the stereographic projection, circles and straight lines are equivalent. Also, orthogonality is preserved.

Points inside the USC transform to points on the southern hemisphere, while points outside transform to the northern hemisphere as illustrated in Fig. 3(b). The two stability circle cases illustrated in Fig. 1(b) and (c) are now represented using stereographic projection to get Fig. 4(a) and (b). In both cases the boundary between the unstable and stable region is defined by a circle in the northern hemisphere. While the situations appear different in the complex plane, they are recognized in the spherical representation as the same. The only difference on the sphere is that in one case the stability circle encloses the north pole. Therefore, topologically it should be possible to determine when the cases represented by Fig. 4(a) and (b) occur using only one parameter. The parameter μ (or μ') does this.

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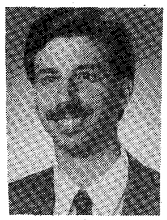

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